Finding fixed points in non-trivial domains: 
Proofs of pending analysis and related algorithms

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Abstract
Young and Hudak\(^1\) developed an algorithm called Pending Analysis for deriving the fixed points of first order functions over binary domains. They prove a result in their paper which justifies the approach, but does not really prove its correctness. Further they suggest that the analysis may be extended to deeper domains. The structure of pending analysis means that many of the intermediate functions generated are not monotonic, which means that arguments of correctness based on partial orderings tend to fail without great care.

This paper attempts to fill some of the gaps in this work, we will:

1. Give proofs of correctness for the binary case when the defining functional is fully monotonic (i.e. monotonic even for non-monotonic function arguments). This is usually for non-self applicatory functions.
2. Develop a slight variant of the algorithm which makes better use of the monotonicity of the function, and which is correct for all functions.
3. Show how the binary analysis can be used to solve more complex domains if the primitives are sufficiently well behaved.
4. Parallel steps 1 and 2 for general domains using an iterative version of the algorithm. In particular proving several incremental fixed point theorems. We will also discus representation and algorithms for the partial functions used in this algorithm.
5. Show that pending analysis (and probably other algorithms too) can be extended to higher order functions by considering the term algebra.
6. Show that the concept of a fixed point can be extended to configurations (partial functions), and that this can be used to generate easy proofs of safeness for pending analysis and similar algorithms.
7. Use these ideas to generate a non-deterministic algorithm which is always correct, and which can be instantiated with different heuristics to yield optimised algorithms. This algorithm includes as special cases pending analysis (with and without memoing) and standard bottom up fixed point calculation.

1. Introduction

1.1. Background - abstract interpretation
Abstract interpretation is widely used technique for analysing programs. Originally developed for flow analysis of imperative languages\(^2\) It has been especially useful in generating information for the efficient compilation of pure functional languages.\(^3,4\)
Examples of abstract interpretation include:

- strictness analysis
- determinism analysis
- type checking
- Currying analysis
- list length analysis

Typically it is the aim of abstract interpretation to completely evaluate the denotation of the program/function/expression in the abstract domain. Again typically, but not necessarily, the abstract domain needs to be finite for this interpretation to succeed for recursive functions. Where the domains are not naturally finite, some adjustment may be necessary. For example, list lengths are unbounded (or even infinite), but analysis can be performed over the domain \{ 0, 1, many \}. Similarly, the domains for Milner type checking are unbounded, but well typed programs have finite types, and careful analysis can prevent infinite recursion for incorrect programs.

Strictness analysis of first order functions is performed over the domain \{ \bot, \top \}, where \bot means definite non-termination and \top means possible termination. Higher order functions have finite domains too, although they grow in size rapidly.

1.2. Solution by fixed points

Having obtained semantic equations for the abstract domain, it is easy to solve those which are non-recursive. Thus abstract interpretation reduces to the finding of fixed points of an abstracted generating functional. The simple solution to this is to iterate, starting at \bot. The problem with this is the large number of argument values we may be solving for. In the first order case there are \(2^n\) possible arguments to a function of arity \(n\), and for higher order functions this increases rapidly. Clack and Peyton-Jones use frontier analysis\(^5\) which is a compact way of storing the values of monotonic functions. This significantly improves the space and time complexity in "average" cases.

1.3. Pending analysis

Unfortunately, even when there are only a few points at which we want to know the functions value, we must iterate until the whole function is stable, as there may be "plateaus" for particular value which may later start to climb again.

Young and Hudak\(^1\) suggest an algorithm called pending analysis which can take advantage of the few values that the fixed point is required at. They suggest that when evaluating the abstracted function \(f\) at the argument value \(a\), we keep a note that \(f(a)\) is "pending", then when we encounter during the expansion of \(f(a)\) another instance of the same application, we substitute \(\bot\) for this. They differentiate this from standard iteration in that it is an "inside-out" strategy whereas pending analysis is "outside-in".

They only present their algorithm for the case of binary domains (with strictness analysis in mind), but do suggest that the technique could be extended to deeper domains by iterating. They also limit themselves to first order functions.

1.4. Purpose and content of paper

Young and Hudak proposed their algorithm after practical experience suggested its correctness. They include a proof, however this is slightly flawed. The purpose of this paper is to give correct proofs of pending analysis for the binary case with conditions for its correctness, and to extend the algorithm to more complex domains, both deeper domains and higher order functions.

Although I have not found any counter examples where simple pending analysis fails, it is only possible to prove its correctness for a limited class of functions, namely non-self applicatory functions. A slightly different algorithm is proposed which is correct for all functions, and which makes better use of the monotonicity of the target function.

Because the flaws in the original proof are quite subtle, and because many obvious properties turn out to be false (or far more difficult than at first glance) the proofs are given in detail, however I hope the progression
from simple to more complex proofs makes this more acceptable. Proofs will give special attention to the
monotonicity of intermediate functions, as this is where most problems arise. They are typically concerned
with two factors:

Safety - giving a result no lower in the domain than is correct.

Optimality - giving a result no higher than necessary.

The safety result is most important in strictness analysis, as it is a more dangerous error to consider an
argument strict that is not than to leave a strict argument lazy. In general we can assume that the domain is ori-
tented such that $\top$ is safe but $\bot$ is most desirable. This is in fact only important when we consider higher
order functions. Young and Hudak make use of the relative importance of safety when they suggest "pes-
simizing", that is returning results deliberately to high in the domain. However "obviously" safe options
such as this must be viewed with extreme care as because of non-monotonicity they may be incorrect
(unsafe).

We will consider first of all, in Section 2, the case of binary domains. The problems in Young and Hudak’s
proof are discussed and correct proofs given, together with the improved algorithm. After this is a short
section giving conditions when simple uniterated pending analysis can be applied to more complex
domains.

Section 4 introduces the representation of partial functions as configurations which will be needed in the
analysis of non-binary domains. Section 5 and 6 give proofs of algorithms for non-binary domains, Section
5 is concerned with non-self applicatory functions, and proves simple pending analysis with iteration cor-
rect for these. Section 6 extends the analysis to all functions using the improved algorithm.

The representation of configurations is crucial to the efficiency of the improved algorithm, and Section 7
discusses ways of structuring and processing these. Higher order functions are discussed in Section 8, and
it is suggested that these may be represented by terms (a reified interpretation) yet still preserve the correct-
ness of the algorithm. Thus saving on the otherwise complex representation of these.

The proofs given are all expressed for evaluation in purely functional terms, and thus optimisations like
memoing require slight reworkings of these, especially to ensure optimality. Instead of doing these directly
Section 9 and 10 take a different approach. Section 9 looks at the fixed point properties of configurations
considered in their own right. These properties suggest a new non-deterministic (or generic) algorithm
which considers configurations as whole. Pending analysis, pending analysis with memoing and standard
iterative fixed point calculation are all instances of this algorithm, and because the non-deterministic algo-
rithm has been proved correct, all instances of it are also. In addition, new instances with more intelligent
heuristics can be formulated.

2. Proofs for binary case

2.1. Notation

For internal consistency with the rest of this paper the notation will differ slightly from that of Young and
Hudak. we are trying to solve for a function $I$ with arguments in a domain $\text{Appl}$ and results in a domain $\mathcal{D}$. That is $I \in \text{Interp}$ where

$$\text{Interp} : \text{Appl} \to \mathcal{D}$$

Typically $I$ represents several user level functions, and its arguments are of the form $f( d_1 \ldots d_n )$ (hence
$\text{Appl}$ - application). Elements of the domain $\text{Interp}$ we will also refer to as interpretations. The desired
function is defined recursively using a functional $G$

$$G : \text{Interp} \to \text{Interp}$$

and is the least fixed point of $G$

$$\text{fix } G = \bigvee I_i \quad \text{where} \quad I_0 = \bot \quad I_{i+1} = G I_i$$

Young and Hudak use the notation $h[ v/\overline{x} ]$ for the function that is equal to $h$ everywhere except at $\overline{x}$
whence it equals $v$. For this section we will only use this when the substituted value is $\bot$, in addition it is
always a function resulting from and application of $G$, thus we will use the notation $G_a$ for the functional such that for all $I$, $G_a I$ equals $G I$ everywhere except at $a$ where it is $\perp$. This is the constrained iterator. We will also use the more general notation $G_{A}$ with the obvious meaning:

$$G_{A} I |_{a} = \begin{cases} \perp & a \in A \\ G I |_{a} & \text{otherwise} \end{cases}$$

$$G_a = G\{a\}$$

The notation $f|_{x}$ is used above to mean the value of $f$ at $x$ (i.e. $f(x)$), and is used to reduce the number of brackets that would otherwise appear. It is consistent with the standard notation (also used) $f|_{S}$ meaning the function identical to $f$ except restricted to the set $S$.

2.2. One pending argument

Young and Hudak give a proof that for arbitrary positive integers, $m, n$ (in their notation)

$$(G^m (G^n(\perp ))(a)) = (G^m (G^n(\perp )[\perp /a ]))(a)$$

or in my notation

$$G^m G^n (\perp )|_{a} = G^m G_{a} G^{n-1}(\perp )|_{a}$$

They give as an informal statement of the theorem

**Theorem**

If, while evaluating $f(a)$ we find that it depends on the value of $f(a)$ again, returning $\perp$ as the result of the second (nested) call to $f(a)$ is correct with respect to the semantics of recursive monotonic boolean functions.

It should be noted that the formal theorem shows that for any such nested call (or to be precise all calls at a certain depth in the call graph) we can substitute bottom, it does not show that we can substitute for all such calls. The formal theorem is thus a justification of pending analysis, rather than a proof of it. Whether the informal and formal statements of the theorem say the same thing is a matter of interpretation.

To prove the stronger case we would need to show

$$G^m G^n (\perp )|_{a} = G^m G_{a} G^{n}(\perp )|_{a}$$

At first sight we could follow exactly the lines of Young and Hudak’s proof for this slightly stronger result:

**Proof**

(1) show $G_{a} I \leq G^n I$

not trivial

(1.1) $G_{0} I = I = G^0 I$

(1.2) induction, assume $G_i I \leq G^i I$

(1.3) $\Rightarrow G G_{a} I \leq G G^i I$

(1.4) $\Rightarrow G_a G_i I \leq G G^i I$

QED 1

(2) $\Rightarrow$ LHS $\geq$ RHS

(3) case LHS = $\perp$ QED
(4) case LHS = ⊤
(5) let \( r \) be the largest integer such that \( G^i(\bot)_a = \bot \)
    show \( G^r_\perp(\bot) = G^r(\bot) \)
(5.1) \( G^0_\perp(\bot) = \bot = G^0(\bot) \)
(5.2) induction, assume \( G^i_\perp(\bot) = G^i(\bot) \quad i < r \)
(5.3) \( \Rightarrow G G^i_\perp(\bot) = G G^i(\bot) \)
(5.4) \( \Rightarrow G G^i_\perp(\bot)_a = \bot = G G^i(\bot)_a \)
(5.5) \( \Rightarrow G_a G^i_\perp(\bot) = G G^i_\perp(\bot) = G G^i(\bot)_a \)

QED 5

(6) show \( G^r_\perp(\bot) \leq G^n_\perp(\bot) \)
    same \( r \) as above
(6.1) \( G^0_\perp(\bot) = \bot \leq G^k_\perp(\bot) \quad \forall k \)
(6.2) induction, assume \( G^i_\perp(\bot) \leq G^j_\perp(\bot) \quad i < j \text{ and } i < r \)
(6.3) \( \Rightarrow G G^i_\perp(\bot) \leq G G^j_\perp(\bot) \)
(6.4) \( \Rightarrow G_a G^i_\perp(\bot) \leq G_a G^j_\perp(\bot) \)

QED 6

(7) \( \Rightarrow G G^i_\perp(\bot) \leq G G^n_\perp(\bot) \)
(8) \( \Rightarrow \top = G G^i(\bot) = G G^i_\perp(\bot) \leq G G^n_\perp(\bot) \)
\( \Rightarrow G G^i_\perp(\bot) = \top \)

QED case LHS = \( \top \)

QED theorem

Now this proof looks fairly convincing, but it is dubious at two places, steps 1.3 and 6.3. Both these steps depend on a monotonicity property of \( G \) of the form:

\( f \leq f' \Rightarrow G f \leq G f' \)

Now this of course is correct for monotonic functions \( f \) and \( f' \), but in both cases we are dealing with the results of \( G_a \) which are of course in general not monotonic. In fact the proof does hold so long as \( G \) is defined using terms (as is the case in Young and Hudak’s work). In this case it satisfies a stronger property it is pseudo-monotonic. That is the above identity hold so long as one or other of \( f \) or \( f' \) is monotonic.

Definition - pseudo-monotonic

\( G \) is pseudo-monotonic if either \( f \) or \( f' \) monotonic and \( f \leq f' \Rightarrow G f \leq G f' \)

If we reexamine the two dubious steps, at 1.3 we find that we are comparing \( G^i_a I \) and \( G^i I \), the later of which is of course monotonic (so long as the seed \( I \) is). In the second case we were comparing \( G^i_\perp(\bot) \) and \( G^j_\perp(\bot) \) however we stipulated that \( i \leq r \), so \( G^i_\perp(\bot) = G^i(\bot) \) which is of course monotonic.

2.3. Further pending arguments

Of course, pending analysis would not be very useful if you could only use it for one argument. It is expected that we would recursively calculate the values of applications using pending analysis as they arise. We can describe this algorithm using the function \( X_\mathcal{A} \) defined recursively as:

\[
X_\mathcal{A}|_a = \bot \quad a \in \mathcal{A} \\
= ( G X_\mathcal{A+\{a\}} |_a ) \quad \text{otherwise}
\]

The final result of pending analysis at a point \( a \) is \( X_{\{a\}}|_a \), and the function \( X_\mathcal{A} \) is the result of pending analysis when we are part way through with \( \mathcal{A} \) pending.

At first glance this seems to hardly merit further proof, if we imagine an example of the form:

\[
f(x, y) = y \quad \text{and} \quad (f(x, x) \text{ or } f(x, y))
\]
Imagine evaluating this at ⊥, T, we would fill in the value of f(⊥, T) as ⊥, then would require the true value of f(⊥, ⊥), this could clearly be obtained by pending analysis itself (given eventually ⊥), hence eventually yielding the correct value.

If we chose a slightly different example however
\[ f(x, y) = y \text{ and } (f(x, x) \text{ or } f(y, x)) \]
This time, we when evaluating at ⊥, T, we would get a call of f(T, T) within the recursive call to f(y, x). in this case we don’t really want the correct value of f(T, T) so much as the correct value when were assuming f(⊥, T) is pending. It seems that it is at least safe to return the correct value, since if anything this is (surely?) as large as the value with an application pending. But is it optimal?

In fact, we shall see that there are more fundamental problems than that, and that the obvious proof, following the line of the above fails in certain cases. We will uncover these by attempting the proof, which turns out to be valid for many cases including the previous examples. By doing this we will uncover an important sub-class of functions that also have a simpler proof for the case of non-binary domains. Further the shape of the proof is very similar to the progressively more complex proofs in the paper and is a useful introduction.

2.4. Proof of binary pending analysis for non-self applicatory functions

We can repeat the proof for single application pending analysis over G, but this time for an embedded case with some applications pending (\(\mathcal{A}\)).

**Theorem**

\[ G_{\mathcal{A}} G_{\mathcal{A}}^n(\bot)_a = G_{\mathcal{A}} G_{\mathcal{A}+(a)}^n(\bot)_a \]

At first sight we could follow exactly the lines of Young and Hudak’s proof for this slightly stronger result:

**Proof**

1. show \( G_{\mathcal{A}+(a)}^n I \leq G_{\mathcal{A}}^n I \)
   
   not trivial

   (1.1) \( G_{\mathcal{A}+(a)}^0 I = I = G_{\mathcal{A}}^0 I \)
   
   (1.2) induction, assume \( G_{\mathcal{A}+(a)}^i I \leq G_{\mathcal{A}}^i I \)
   
   (1.3) \( \Rightarrow G G_{\mathcal{A}+(a)}^i I \leq G G_{\mathcal{A}}^i I \)
   
   (1.4) \( \Rightarrow G_{\mathcal{A}+(a)} G_{\mathcal{A}+(a)}^i I \leq G_{\mathcal{A}} G_{\mathcal{A}}^i I \)
   
   **QED 1**

2. case LHS \(\geq\) RHS

3. case LHS = \(\bot\) **QED**

4. case LHS = T

5. let \(r\) be the largest integer such that \( G_{\mathcal{A}}(\bot)_a = \bot \)

   show \( G_{\mathcal{A}+(a)}(\bot)_a = G_{\mathcal{A}}(\bot)_a \)

   --- works as before ---

6. show \( G_{\mathcal{A}+(a)}(\bot)_a \leq G_{\mathcal{A}+(a)}^n(\bot)_a \)

   same \(r\) as above

   (6.1) \( G_{\mathcal{A}+(a)}^0(\bot)_a = \bot \leq G_{\mathcal{A}}^k(\bot)_a \forall k \)

   (6.2) induction, assume \( G_{\mathcal{A}+(a)}^i(\bot)_a \leq G_{\mathcal{A}+(a)}^j(\bot)_a \)

   (6.3) \( \Rightarrow G G_{\mathcal{A}+(a)}^i(\bot)_a \leq G G_{\mathcal{A}+(a)}^j(\bot)_a \)

   (6.4) \( \Rightarrow G_{\mathcal{A}+(a)} G_{\mathcal{A}+(a)}^i(\bot)_a \leq G_{\mathcal{A}+(a)} G_{\mathcal{A}+(a)}^j(\bot)_a \)

   **QED 6**

7. \( \Rightarrow G_{\mathcal{A}} G_{\mathcal{A}+(a)}(\bot)_a \leq G_{\mathcal{A}} G_{\mathcal{A}+(a)}^n(\bot)_a \)

8. \( \Rightarrow T = G_{\mathcal{A}} G_{\mathcal{A}}(\bot)_a = G_{\mathcal{A}} G_{\mathcal{A}+(a)}(\bot)_a \leq G_{\mathcal{A}} G_{\mathcal{A}+(a)}^n(\bot)_a \)

   \( \Rightarrow G_{\mathcal{A}} G_{\mathcal{A}+(a)}^n(\bot)_a = T \)

   **QED** case LHS = T

**QED** theorem
Step 5 is an argument about equality and is fine, however steps 1.3 and 6.3 are there to worry us again. The pseudo-monotonicity of \( G \) (and likewise \( G^A \)) is of no avail, as in general neither of \( G_{A+\{a\}} I \) and \( G^I \) are monotonic even when \( I = \bot \) (the particular case of interest). Similarly in step 6.3 neither \( G_{A+\{a\}}^I( \bot ) \) nor \( G^A_{I+\{a\}}( \bot ) \) will be monotonic. In fact it is quite on the cards that if a functional \( F \) does not yield monotonic results then we can have \( F^{n+1}( \bot ) \leq F^n( \bot ) \).

If we try out various examples (such as those in the previous section) we find that in many cases these properties do hold, and the temporary non-monotonic functions are acceptable.

If we imagine iterating for a typical function, say \( f \)

\[
f^*(x, y) = y \text{ and } (f(x, x) \text{ or } f(y, x))
\]

where \( f^* \) means the next iteration for \( f \). We find that if \( f \leq f' \) then \( f^* \leq f'^* \) whether or not \( f \) and \( f' \) are monotonic. That is the generating functional for \( f \) is monotonic for all functions whether monotonic or not. We will call such a functional \textit{fully monotonic}.

\textbf{Definition - fully monotonic}

\[ G \text{ is fully monotonic if } \forall f, f' \quad f \leq f' \Rightarrow Gf \leq Gf' \]

\( f \) and \( f' \) not necessarily monotonic

When do the "general" cases, where the proof fails, arise? The answer is self applicatory functions. That is functions defined so that an argument to the function may contain an expression dependent on the function.

For instance, \( \text{fib} \) is non-self applicatory

\[
\text{fib}(n) = \begin{cases} 1 & \text{if } n < 2 \\ \text{fib}(n-1) + \text{fib}(n-2) & \text{else} \end{cases}
\]

In the definition of \( \text{fib} \), the arguments to it are \( n-1 \) and \( n-2 \), which do not contain references to \( \text{fib} \).

On the other hand \( \text{tak} \) is self applicatory

\[
\text{tak}(x, y, z) =
\begin{cases}
  z & \text{if } x < y \\
  \text{tak}(x-1, y, z), & \text{else} \\
  \text{tak}(y-1, z, x), & \text{tak}(z-1, x, y)
\end{cases}
\]

The resulting functional is not fully monotonic (although of course still pseudo-monotonic).

It is easy to see where a loss of monotonicity can appear. If we increase the input value of \( f \) to an expression like \( f(f(\top, \top), \bot) \), then the result of the inner application of \( f \) will clearly increase, but if \( f \) was not monotonic then the outer application may decrease. For example:

\[
\begin{align*}
f(\bot, \bot) &= \top \\
f(\top, \bot) &= \bot \\
f(\top, \top) &= \bot \\
\end{align*}
\]

so \( f \leq f' \) but

\[
\begin{align*}
f(f(\top, \top), \bot) &\rightarrow f(\bot, \bot) \rightarrow \top \\
f(f(\top, \top), \bot) &\rightarrow f(\top, \bot) \rightarrow \bot \\
\end{align*}
\]

Formally, a first-order term is self-applicatory if every application of a user defined function is a constant term (that is a term involving primative operations and arguments only). When we are defining several user defined functions, we can treat non-mutually recursive functions as primative. The higher order case is slightly more complex as we need also be careful of non-fully monotonic primatives (like application). Self application is easy to check statically, and could be done as a pre-pass before more extensive analysis, in order to select appropriate algorithms.

If we restrict ourselves to functionals defined by non-self applicatory terms, we can show that they are fully monotonic (see appendix) and thus the above proof holds. We can thus prove binary pending analysis
correct for this case.  
The general result proved is that \( X_\mathcal{A} = \text{fix } G_\mathcal{A} \), which of course includes as a special case \( X_\{\} = \text{fix } G \).  
Because the \( G_\mathcal{A} \) are fully monotonic, the fixed point construction makes sense even though the resulting functions are not monotonic. It is useful to restate the result of the above theorem as:

**Incremental fixed point theorem for binary functions**

\[
\text{fix } G_{\mathcal{A}}|_a = G_{\mathcal{A}} \text{ fix } G_{\mathcal{A} + \{a\}}|_a
\]

Further on we shall prove two similar results, one for non-binary but self applicatory functionals, and one for general functionals. These results will however become progressively more complex. We can thus prove the correctness. When \( \mathcal{A} \) is complete then \( X_\mathcal{A}, G_\mathcal{A} \) and hence \( \text{fix } G_\mathcal{A} \) are all identically \( \perp \). The recursive case is proved using the above theorem.

**Theorem**

\( X_\mathcal{A} = \text{fix } G_\mathcal{A} \)

**Proof**

Induction on size \( \mathcal{A} \).

Base case \( \mathcal{A} \) complete

\[
X_\mathcal{A} = \perp \\
G_\mathcal{A} = \perp \Rightarrow \text{fix } G_\mathcal{A} = \perp
\]

**QED** base case

Inductive case.

Sub-cases

case \( a \in \mathcal{A} \)

\[
X_{\mathcal{A} \setminus a} = \perp = G_{\mathcal{A}} X_{\mathcal{A} + \{a\}}|_a
\]

**QED** \( a \in \mathcal{A} \)

case \( a \notin \mathcal{A} \)

\[
X_{\mathcal{A} \setminus a} = G_{\mathcal{A}} X_{\mathcal{A} + \{a\}}|_a \\
= G_{\mathcal{A}} \text{ fix } G_{\mathcal{A} + \{a\}}|_a \quad \text{- induction} \\
= \text{fix } G_{\mathcal{A} \setminus a} \quad \text{- incremental fixed point theorem}
\]

**QED** \( a \notin \mathcal{A} \)

QED inductive cases

**QED** Theorem

Again we shall see similar proofs later on for non-binary self applicatory and for general functionals.

### 2.5. General, binary functionals

Simple pending analysis does appear to still work in these cases, but proofs break down. In order to produce a proof, we must alter the algorithm slightly. The problem is having intermediate non-monotonic functions. For the non-self applicatory case this did not matter as the functionals were fully monotonic. In the general case we cannot rely on this and must therefore force the intermediate results to be monotonic. The simplest way to do this is to always extend the pending values in \( \mathcal{A} \) so as to make it downwards complete, that is:

\[ a \in \mathcal{A} \quad \text{and} \quad a' < a \Rightarrow a' \in \mathcal{A} \]

If this is so, then \( G_{\mathcal{A}} \) will take monotonic functions to monotonic functions, hence all the intermediate results are monotonic. The only extra problem is that we will add more than one element to \( \mathcal{A} \), does this mean we have to iterate several times? In fact not, but we have to modify the proofs very slightly to take account of this.

The new algorithm is:
The sets $A$ can be proved exactly as the non-self applicatory case, except that the recursive case has $A$ where to $A$.

We proceed by proving and incremental fixed point theorem, and then showing that $Z_A = \text{fix } H_A$. All the sets $A$ considered are downwards closed. That is:

$$A \oplus \{a\} = A \cup \{a' \mid a' \leq a\}$$

The proof given in Young and Hudak’s paper is useful in convincing one of the correctness of pending analysis applied to one application, but even this uses a property of functionals defined using terms, pseudo-monotonicity, the normal monotonicity of the functional being insufficient to complete the proof. The case of multiple pending applications is more difficult and we needed to restrict ourselves to non-self applicatory functions when the defining functional would be fully-monotonic and hence the non-

\[ Z_{A_a} = \perp \quad a \in A \]
\[ = (G Z_{A \oplus \{a\}})_a \quad \text{otherwise} \]

where $A \oplus \{a\}$ is $A$ extended by $A$ and then downwards closed. That is:

$$A \oplus \{a\} = A \cup \{a' \mid a' \leq a\}$$

Theorem - Incremental fixed point theorem for general binary functions

$$\text{fix } G_{A, a} = G_A \text{ fix } G_{A \oplus \{a\}, a} \quad - A \text{ downwards closed}$$

Proof

Steps -

(1) $G_A \geq G_{A \oplus \{a\}}$
(2) $\Rightarrow$ LHS $\geq$ RHS
(3) case LHS = $\perp$ QED
(4) case LHS = $\top$
(5) let $r$ be largest such that $G_{A, i}(\perp)_a = \perp$
  show $G_{A \oplus \{a\}, i}(\perp) = G_{A, i}(\perp)$
(5.1) $G_{A \oplus \{a\}, i}(\perp) = \perp = G_{A, i}(\perp)$
(5.2) induction, assume $G_{A \oplus \{a\}, i}(\perp) = G_{A, i}(\perp)$, $i < r$
(5.3) $\Rightarrow G_{A \oplus \{a\}, i}(\perp) = G_{A, i}(\perp)$
(5.4) $\Rightarrow G_{A \oplus \{a\}, i}(\perp)_a = \perp = G_{A, i}(\perp)_a$
(5.5) $\forall a' \leq a \quad G_{A \oplus \{a\}, i}(\perp)_a = \perp$ - monotonicity
(5.6) $\Rightarrow G_{A \oplus \{a\}, i} G_{A \oplus \{a\}, i}(\perp) = G_A G_{A \oplus \{a\}, i}(\perp) = G_A G_{A \oplus \{a\}, i}(\perp)$
QED 5
(6) show $G_{A \oplus \{a\}, i}(\perp) \leq G_{A \oplus \{a\}, i}(\perp)$
  same $r$ as above
  ---- proof exactly as for $G_{A, i} ----$
(7) $\Rightarrow G_A G_{A \oplus \{a\}, i}(\perp) \leq G_A G_{A \oplus \{a\}, i}(\perp)$
(8) $\Rightarrow \top = G_A G_{A \oplus \{a\}, i}(\perp) = G_A G_{A \oplus \{a\}, i}(\perp) \leq G_A G_{A \oplus \{a\}, i}(\perp)$
  $\Rightarrow G_A G_{A \oplus \{a\}, i}(\perp) = \top$
(9) $\Rightarrow G_A \text{ fix } G_{A \oplus \{a\}, i} = \top$
QED case LHS = $\top$

QED theorem

The only difference from the previous proof was step 5.5, where we had to be careful that all the additions to $A \oplus \{a\}$ were already $\perp$, but the monotonicity of the intermediate results ensured this. The final correctness of the algorithm, namely:

Theorem - correctness of pending analysis for general binary functions

$$Z_A = \text{fix } G_A \quad - A \text{ downward closed}$$

can be proved exactly as the non-self applicatory case, except that the recursive case has $A \oplus \{a\}$ rather than $A + \{a\}$. This proof is omitted.

2.6. Discussion of binary case

The proof given in Young and Hudak’s paper is useful in convincing one of the correctness of pending analysis, but does not prove it entirely. It is possible to use the form of their proof to give a full proof of correctness of pending analysis applied to one application, but even this uses a property of functionals defined using terms, pseudo-monotonicity, the normal monotonicity of the functional being insufficient to complete the proof. The case of multiple pending applications is more difficult and we needed to restrict ourselves to non-self applicatory functions when the defining functional would be fully-monotonic and hence the non-
monotonicity of the intermediate terms acceptable. Finally, we proved the case of general binary functions by modifying the basic algorithm slightly in order to ensure monotonicity of intermediate results. We have not proved the general case using simple pending analysis, but equally I have not found any counter examples, although there are functionals that do not satisfy apparently sensible properties like:

\[ G \, G_{\mathcal{A}} \, \text{fix} \, G \leq G \, G_{\mathcal{A}} \, \text{fix} \, G \]

3. Extending binary analysis by abstract interpretation

If we are working over a non-binary domain \( D \) we may still be able to use the results for the binary case if the primitive operations have a particular form. For any \( y \) in \( D \) define:

\[
\begin{align*}
\text{Abs}_y : D & \rightarrow \mathbb{2} \\
d_y & = \top \quad \text{if} \quad d \geq y \\
& = \bot \quad \text{otherwise}
\end{align*}
\]

Now if for each \( y \) and each primitive operator \( \mu \) we can define an abstract version \( \mu_y \) such that:

\[ \forall a \in \text{dom} \mu \quad \mu_y(a_y) = (\mu(a))_y \]

Then this identity is also satisfied for the functional \( G \) defined in terms of them, and we can prove that simple pending analysis also works for this non-binary case.

Sketch of proof

Assume that \( \text{fix} \, G(a) = y \). We use \( \text{Abs}_y \) and find that the result of the abstract pending analysis is \( \top \), and hence show that the result of pending analysis is at least as big as \( y \). We then by consider \( \text{Abs}_y \) for all \( y' \) above \( y \), and similarly show that the abstract pending analysis yields \( \bot \) and hence that the result of pending analysis is less than all these \( y' \).

In the following, let \( \text{Pend}(F, x) \) be the result of applying pending analysis to obtain the value of the fixed point of \( F \) at the point \( x \) in the appropriate domain.

Proof

\[
\begin{align*}
\text{Pend}(G, a) = y & \Rightarrow \text{Pend}(G, a) \geq y \\
\text{fix} G(a) & = y \\
\text{fix} G_y(a_y) & = \top \\
\text{Pend}(G_y, a_y) & = \top \quad \text{- correctness of pending analysis for binary domain} \\
(Pend(G, a))_y & = \top \quad \text{- pending analysis uses only primitive operations} \\
\text{Pend}(G, a) & \geq y \quad \text{- def’n of Abs}_y \\
QED \quad 1
\end{align*}
\]

Step 2 - prove \( y < y' \Rightarrow \text{Pend}(G, a) < y' \)

\[
\begin{align*}
\text{Pend}(G, a) = y & \Rightarrow \text{Pend}(G, a) < y' \\
\text{fix} G(a) & = y \\
\text{fix} G_y(a_y) & = \bot \quad y < y' \\
\text{Pend}(G_y, a_y) & = \bot \\
(Pend(G, a))_y & = \top \\
\text{Pend}(G, a) & < y' \quad \text{- def’n of Abs}_y \\
QED \quad 2
\end{align*}
\]

4. Configurations

Frontier algorithms and the like are based around producing successive interpretations approximating \( \text{fix} \, G \). That is they find values for all possible function applications simultaneously. Pending analysis seeks to only look at sufficient points to find the value of \( \text{fix} \, G \) for a particular function at a particular point. We will therefore want to consider partial interpretations. We will call a partial interpretation a configuration.
The set of configurations is then a subset of the partial functions between applications and results.

\[ \text{Config} \subseteq \text{Appl} \rightarrow \mathcal{D} \]

We will use the notation \( C + (a \rightarrow d) \) for the extension of \( C \) to map an application \( a \) to a result \( d \). It is assumed that \( a \notin \text{dom} C \). That is:

**Definition** - \( C + (a \rightarrow d) \)

\[
\begin{align*}
\text{dom} C + (a \rightarrow d) &= \text{dom} C + \{a\} \\
C + (a \rightarrow d)|_{\text{dom} C} &= C \\
C + (a \rightarrow d)|_{a} &= d
\end{align*}
\]

We will use several partial orderings on configurations.

We can compare configurations using a pointwise comparison:

\[ C \leq C' \equiv \begin{cases} \text{dom } C = \text{dom } C' \\ \forall a \in \text{dom } C, \ C(a) \leq C'(a) \end{cases} \]

We will also need a domain oriented ordering -

\[ C \subset C' \equiv \begin{cases} \text{dom } C \subset \text{dom } C' \\ \forall a \in \text{dom } C, \ C(a) = C'(a) \\
\text{i.e. } C|_{\text{dom } C} = C \end{cases} \]

Also we want a mixture of the two:

\[ C \ll C' \equiv \begin{cases} \text{dom } C \subset \text{dom } C' \\ \forall a \in \text{dom } C, \ C(a) \leq C'(a) \end{cases} \]

\[ C \gg C' \equiv \begin{cases} \text{dom } C \subset \text{dom } C' \\ \forall a \in \text{dom } C, \ C(a) \geq C'(a) \end{cases} \]

These last two relations are weaker than the above two:

\[ C \ll C \iff C \ll C' \text{ and } C' \gg C \]

\[ C \gg C \iff C \gg C' \text{ and } C' \ll C \]

\[ C \subset C \iff C \ll C' \text{ and } C' \gg C \]

A configuration is **total** if its domain is all of \( \text{Appl} \).

A configuration (partial or total) is said to be **monotonic** if:

\[ \forall a, a' \in \text{Appl} \quad a \leq a' \Rightarrow C_a \leq C_a' \]

A total configuration can be regarded as an interpretation and this definition of monotonicity is then standard monotonicity.

We can express the recursion as a function \( H \) over total monotonic configurations:

\[
H : \text{Config} \rightarrow \text{Config}
H C = \cap \{ G I \mid C \subseteq I \}
\]

Note that this is the intersection of the \( G I \), that is the result is a configuration which is only defined at the points where all the \( I \) agree. An equivalent definition would be.

\[
\begin{align*}
\text{dom} (H C) &= \{ a \mid \forall I, I' \quad C \subseteq I \quad \text{and} \quad C \subseteq I' \Rightarrow G I(a) = G I'(a) \} \\
(H C)|_{a} &= G I(a) \text{ for any } I \quad \text{st} \quad C \subseteq I
\end{align*}
\]

\( H \) is thus clearly a monotonic (wrt. \( \ll, \gg, \leq \) and \( \subset \)) continuous functional over monotonic configurations and equal to \( G \) for total configurations. In most of the following we will only apply \( H \) to total configurations, but we will use \( H \) consistently whether the configuration is total (and \( G \) would do) or not. Typically
when a configuration is total we will refer to it as I or J, possibly primed, whereas configurations will be $\mathcal{C}$, $\mathcal{C}'$ etc.

5. Pending analysis for non-self-applicatory functions

Young and Hudak suggest that their algorithm can be extended to non-binary domains by iterating the pending analysis at each stage. This brings to mind numerical optimisation algorithms, where when optimising several quantities, one fixes one of them and optimises over the rest under that assumption. Finally one optimises the chosen "pivot" value using the result of the recursive optimisation as ones target function.

In the binary case, we had the simple recursion equation for $X_A$, the function "given" that the elements of $\mathcal{A}$ are $\perp$.

$$X_{\mathcal{A}} |_a = \perp\quad a \in \mathcal{A}$$
$$X_{\mathcal{A}} |_a = H \cdot X_{\mathcal{A} \rightarrow a} |_a\quad \text{otherwise}$$

This is more complex when the domain is deeper. Whereas the in the binary case we get the right value after one application of $H$ we must either expand the function deeper, or iterate. By expanding several calls before we apply the pending value, we would do too much work if the value were quite low in the domain, and thus the second course seems better. Iterating requires relatively minor changes to the binary algorithm. In the binary case, an application was either not pending or $\perp$, in the non-binary case an application may be pending with various values and hence we use a configuration to capture these pending values.

When we come to expand the value of a function, and the application is pending, we return the pending value as held in the configuration:

$$X_{\mathcal{L} \mathcal{C}} |_a = \mathcal{C}(a)\quad a \in \text{dom} \mathcal{C}$$

When we come to expand the value of a function, we must change the rule when the application is not pending. For binary analysis this was:

$$X_{\mathcal{L} \mathcal{A}} |_a = H \cdot X_{\mathcal{L} \mathcal{A} \rightarrow a} |_a$$

this must now include a fixed point calculation:

$$X_{\mathcal{L} \mathcal{C}} |_a = \text{fix } h$$

where

$$h(\ d\ ) = (\ H \cdot X_{\mathcal{L} \mathcal{C} \rightarrow (a \rightarrow d)} |_a)$$

Clearly we must again be careful about non-monotonic intermediate results, as otherwise the fixed point calculation may not be valid and we cannot argue sensibly about $\leq$ conditions. Thus we will first confine ourselves to the case of non-self applicatory functionals as we did for binary functions. We will return to the general case in the next section with a slightly modified algorithm. Non-self applicatory functions are ok because their generating functionals are fully monotonic. Although intermediate results will be non-monotonic the final function is monotonic however.

In order to prove the algorithm correct we will need to prove an incremental fixed point theorem that says we can calculate the fixed point of $H$ at a point $a$ by first working out a fixed point of a function using $H$ fixed at various values. In general we will have that the fixed point of $H^\mathcal{C}_a$ ($H$ fixed to take the values of $\mathcal{C}$) can be expressed in terms of $H^\mathcal{C}_{\mathcal{L} \mathcal{C} \rightarrow (a \rightarrow d)}$:

**Incremental fixed point theorem for fully monotonic functionals**

$$\text{fix } H^\mathcal{C}_a |_a = \text{fix } g$$

where

$$g(\ d\ ) = H^\mathcal{C}_a (\ \text{fix } H^\mathcal{C}_{\mathcal{L} \mathcal{C} \rightarrow (a \rightarrow d)}) |_a$$

$H^\mathcal{C}_a$ can be defined in terms of a limiting functional $\mathcal{L}^\mathcal{C}_a$. 
Clearly \( L(\) is the identity and thus \( \text{fix } H^L = \text{fix } H \). Further \( L \) is fully monotonic so all the \( H^L \), by composition with \( H \), will be also.

Given this we can prove the correctness of the evaluation algorithm.

\[
\begin{align*}
H_C^L &= \ L_C^H \\
L_C^F \mid a &= C(a) \quad a \in \text{dom } C \\
&= F(a) \quad \text{otherwise}
\end{align*}
\]

We want \( X_L^C \) to be equal to \( \text{fix } H^L \), so we prove the general case that for all \( C \), \( X_L^C = \text{fix } H^L \). The incremental fixed point theorem is used to prove the recursive case. We proceed by induction on the size of \( C \).

### 5.1. Correctness of algorithm for non-self applicatory terms

**Theorem**

\( X_L^C = \text{fix } H_C^L \)

**Proof**

Induction on size of \( C \)

We assume that \( X_L^C = \text{fix } H_C^L \), for any \( C \) such that \( \| C \| > C \)

**Base case** \( C \text{ total} \)

In this case \( H_C^L \) is constantly \( C \) and thus \( \text{fix } H_C^L = C \)

Similarly \( X_L^C \) is constantly \( C \)

QED \( C \text{ total} \)

Prove \( \forall a \quad X_L^C (a) = (\text{fix } H_C^L)_a \)

There are two cases:

- either \( a \in \text{dom } C \)
  - \( X_L^C (a) \to C_a \)
  - but \( \text{fix } H_C^L_{\text{dom } C} = C \)
  - QED \( a \in \text{dom } C \)

- or \( a \notin \text{dom } C \)
  - \( X_L^C (a) \to \text{fix } h \)
    - where \( h(d) = H \times X_L^C (a \to d)_a \)
  - now \( C + (a \to d) \parallel \geq \parallel C \parallel \)
  - \( \Rightarrow H X_L^C (a \to d) = H \text{fix } H_C^L (a \to d) \) - by induction
  - That is \( h \) here is equal to \( g \) of the incremental fixed point theorem, and as \( g \) is monotonic the \( \text{fix } h \) is the minimal fixed point of \( g \).
  - We thus apply the theorem :-
    - \( X_L^C (a) = \text{fix } g = (\text{fix } H_C^L)_a \)
    - QED \( a \notin \text{dom } C \)

QED inductive cases

**QED** theorem

### 5.2. Incremental fixed point theorem for fully monotonic functionals.

As the theorem includes the fixed point of \( g \) it is slightly different from the proofs for the binary case. First of all we prove that \( d_{\text{fix}} \leq (\text{fix } H_C^L)_a \). Rather than trying to prove the reverse equality directly, we use
this result to prove more generally that and then use this to show that in general \( \text{fix } H^L_C \geq \text{fix } H^L_{C_{fix}} \).

Finally, we prove the reverse inequality for the general case by showing that \( \text{fix } H^L_{C_{fix}} \) is a fixed point of \( H^L_C \).

**Theorem - incremental fixed point theorem**

Let \( g(a) = (H^L_C(\text{fix } H^L_{C+(a \rightarrow d)})a \)

and \( d_{fix} = \text{fix } g \)

and \( C_{fix} = C + (a \rightarrow d_{fix}) \)

then

(i) \( (\text{fix } H^L_C)_a = d_{fix} \)

(ii) \( \text{fix } H^L_C = \text{fix } H^L_{C_{fix}} \)

**Proof**

Steps -

(1) show that \( d \leq d_{lhs} \Rightarrow g(d) \leq d_{lhs} \)

where \( d_{lhs} = (\text{fix } H^L_C)_a \)

Proof step 1

(1.1) \( d \leq d_{lhs} \Rightarrow \text{fix } H^L_{C|\text{dom } C+(a \rightarrow d)} \geq C + (a \rightarrow d) \)

(1.2) \( \Rightarrow H^L_{C+(a \rightarrow d)} \text{fix } H^L_C \leq \text{fix } H^L_C \)

(1.3) \( \Rightarrow \text{fix } H^L_{C+(a \rightarrow d)} \leq \text{fix } H^L_C \)

(1.4) \( \Rightarrow H^L_C \text{fix } H^L_{C+(a \rightarrow d)} \leq H^L_C \text{fix } H^L_C = \text{fix } H^L_C \)

(1.5) \( \Rightarrow \text{fix } H^L_C \leq d(d) \leq d_{lhs} \)

QED (1)

(2) \( \Rightarrow d_{fix} = \text{fix } g \leq d_{lhs} \)

(3) \( \Rightarrow \text{fix } H^L_{C_{fix}} \leq \text{fix } H^L_C \)

(4) let \( I = \text{fix } H^L_{C_{fix}} \)

prove that \( H^L_C I = I \)

(4.1) \( (H^L_C I)_a = d_{fix} \)

- immediate from def'n of \( g \) and \( g(d_{fix}) = d_{fix} \)

(4.2) \( a \neq a \Rightarrow (H^L_C I)_a = (H^L_{C_{fix}} I)_a \)

QED (4)

(5) thus \( \text{fix } H^L_{C_{fix}} = I \geq \text{fix } H^L_C \) - minimality of \( \text{fix } H^L_C \)

QED (ii)

(6) \( \Rightarrow (\text{fix } H^L_{C_{fix}})_a = g(d_{fix}) = d_{fix} \)

QED (i)

QED theorem

Note the various properties of \( H^L_C \) that have been used through this proof.

Step 1.2 uses the fixing property of \( L_C \), that is:

\[ H_{\text{dom } C} \geq (\varnothing, \varnothing) \Rightarrow I \geq (\varnothing, \varnothing) \]

and the absorption property of \( L_C \) and \( H^L_C \)

\[ L_C + (a \rightarrow d) \circ L_C = L_C + (a \rightarrow d) = L_C \circ L_C + (a \rightarrow d) \]

\[ \Rightarrow H^L_C + (a \rightarrow d) = L_C + (a \rightarrow d) \circ H^L_C \]

We will need a similar result for the slightly different iterating function which we will use later for general functions.

Step 1.3 uses the property of fixed points

\[ F f \leq f \Rightarrow \text{fix } F \leq f \]

and requires that \( H^L_{C+(a \rightarrow d)} \) be monotonic.
6. Pending analysis for general functions

6.1. Need for a bounding function

As we’ve said above, not all function definitions are free of self application hence we must produce a more complex analysis to deal with this case. We will need to force interpretations to agree with the assumed configuration in a way that leaves them monotonic. We must however be very careful in the way we do this however. We might consider using the ceiling function to normalise interpretations (where ceiling( I ) is the least monotonic interpretation greater then or equal to I). This however implies a global analysis of the interpretation and is not suitable for a pending analysis algorithm. So we must beware on two points

- Montonicity - intermediate results must be monotonic
- Locality - applying the result of a functional at a point must depend on just a few values of the functionals argument.

The incremental fixed point theorem below instead makes use of a "bounding" function that constrains interpretations to agree with a configuration but does not use any information except the value of the interpretation at the point of interest, and the configuration itself, satisfying both constraints. This leads to an effective (but inefficient) algorithm. However, having proved the basic algorithm correct, various improvements are apparent.

We will have to prove properties of the bounding function \( \beta_c \) similar to those required for the limiting function \( \Lambda_c \) in the previous section, such as the functions which are its fixed points, absorption properties. These will then be used to prove properties of the constrained iterator \( H_c = \beta_c H \) necessary to prove an iterative fixed point theorem which should by now be familiar. This will be used to prove the correctness of a variant of pending analysis in a proof which again should be familiar!

In general all the results proved seem obviously true, however some need surprising care, hence the slow build up from properties of configurations, to those of the bounding function, those the iterator and then finally the fixed point theorem itself.

6.2. Configuration operators

Before describing the bounding function itself, we need a few new operators on configurations. For any configuration \( \mathcal{C} \) we define \( \mathcal{C}^\top \) and \( \mathcal{C}^\bot \) as follows:

\[
\begin{align*}
\mathcal{C}^\top(a) & = \bigwedge_{a' \geq a} \mathcal{C}(a') \\
\mathcal{C}^\bot(a) & = \bigvee_{a \leq a'} \mathcal{C}(a')
\end{align*}
\]

We can think of \( \mathcal{C}^\top \) as the greatest total monotonic configuration below \( \mathcal{C} \), and similarly \( \mathcal{C}^\bot \) is the least above \( \mathcal{C} \). That is:

for all total and monotonic \( \mathcal{C}' \),
\[
\begin{align*}
\mathcal{C} & \gg \mathcal{C}' & \Rightarrow & \mathcal{C}^\top & \geq & \mathcal{C}' \\
\mathcal{C} & \ll \mathcal{C}' & \Rightarrow & \mathcal{C}^\bot & \leq & \mathcal{C}'
\end{align*}
\]

These operators preserve some of the ordering properties of the configurations.
Lemma

(i) \( C \ll C' \Rightarrow C^\perp \leq C'^\perp \)
(ii) \( C \gg C \Rightarrow C^\top \geq C'^\top \)

Proof

The two results are duals so we prove the former
\[
C^\perp = \bigvee_{a \leq a'} C(a') \\
\leq \bigvee_{a \leq a' \text{ and } a \in \text{dom } C} C(a') \\
\leq \bigvee_{a \leq a'} C(a') \\
= C'^\perp
\]

\( QED \) (i)

QED lemma

Lemma

\( C \subseteq C' \Rightarrow C^\perp \leq C'^\perp \) and \( C^\top \leq C'^\top \)

Proof

\[
C \subseteq C' \quad C \ll C' \quad \text{and} \quad C' \gg C \\
\Rightarrow C^\perp \leq C'^\perp \quad \text{and} \quad C'^\top \geq C^\top \\
\quad \text{– applying (i) and (ii) above}
\]

QED lemma

Lemma

\( C \subseteq C' \Rightarrow C^\perp \leq C'^\perp \) and \( C^\top \geq C'^\top \)

Proof

\[
C \subseteq C' \quad C \ll C' \quad \text{and} \quad C' \gg C \\
\Rightarrow C^\perp \leq C'^\perp \quad \text{and} \quad C'^\top \geq C^\top \\
\quad \text{– applying (i) and (ii) above}
\]

QED lemma

Lemma

\( C \) monotonic \( \Rightarrow C^\top|_{\text{dom } C} = C \)

\( C^\perp|_{\text{dom } C} = C \)

\( C^\perp \leq C^\top \)

Proof

first two trivial.
last simple too -

\[
\forall a, a', a'' \in \text{Appl} \quad \text{st} \quad a' \leq a \leq a'' \\
\quad a', a'' \in \text{dom } C \Rightarrow C(a') \leq C(a'') \\
\quad \text{– monotonicity of } C
\]

thus

\[
\bigwedge_{a \leq a'} C(a') \leq \bigvee_{a' \geq a} C(a'')
\]

that is \( C^\perp(a) \leq C^\top(a) \) as required

QED lemma

The eventual algorithm will work by incrementally adding to the set of assumptions. As we need to retain a monotonic configuration we need to add in such a way as to preserve monotonicity.
Definition - $\mathcal{C} \oplus (a \rightarrow d)$

$$\text{dom } \mathcal{C} \oplus (a \rightarrow d) = (\text{dom } \mathcal{C}) + \{a\}$$

$$\mathcal{C} \oplus (a \rightarrow d)_{\text{dom } \mathcal{C}} = \mathcal{C}$$

$$\mathcal{C} \oplus (a \rightarrow d)(a) = (d \land \mathcal{C}^T(a)) \lor \mathcal{C}^\bot(a)$$

Lemma

$\mathcal{C}$ monotonic $\Rightarrow \mathcal{C} \oplus (a \rightarrow d)$ also monotonic

$\mathcal{C} \subseteq \mathcal{C} \oplus (a \rightarrow d)$

$\mathcal{C} \oplus (a \rightarrow d)$ is monotonic ($\subseteq$) as a function of $d$

Proof - trivial

6.3. The bounding function

Now we can define the bounding function in terms of $\mathcal{C}^T$ and $\mathcal{C}^\bot$:

Definition

$$\mathcal{B}_C(I) = (I \land \mathcal{C}^T) \lor \mathcal{C}^\bot$$

For monotonic $\mathcal{C}$ the order of bounding is not important as (the above lemma) $\mathcal{C}^\bot \leq \mathcal{C}^T$. Note also that $\mathcal{B}_C$ has good locality properties, in order to work out $(\mathcal{B}_C(I))_a$ one only needs to know the value of $I$ at $a$ and the value of the configuration.

We prove several lemmas, the first three regarding the fixed points of $\mathcal{B}_C$, and after that an absorption property. These will be used in the same way as the corresponding properties for $\mathcal{L}_C$ in the non-self applicatory case. Finally we will prove the necessary monotonicity properties of $\mathcal{B}_C$.

Lemma - fixing

$\mathcal{C}$ monotonic $\Rightarrow \forall I \mathcal{B}_C(I)_{\text{dom } \mathcal{C}} = \mathcal{C}$

Proof

let $I' = \mathcal{B}_C(I)$

$$\forall a \in \text{dom } \mathcal{C}$$

$$\mathcal{C}^T(a) = \mathcal{C}(a) \text{ and } \mathcal{C}^\bot(a) = \mathcal{C}(a)$$

$$\Rightarrow$$

$$I'(a) = (I(a) \land \mathcal{C}^T(a)) \lor \mathcal{C}^\bot(a)$$

$$= \mathcal{C}(a)$$

QED lemma
Lemma - fixed points of $B_C$

$\forall \ C$ monotonic and $I$ monotonic

(i) $I_{\text{dom } C} \supseteq C \Rightarrow B_C(I) \leq I$
(ii) $I_{\text{dom } C} \subseteq C \Rightarrow B_C(I) \geq I$
(iii) $I_{\text{dom } C} = C \Rightarrow B_C(I) = I$

Proof

Clearly (i) + (ii) $\Rightarrow$ (iii)

Also (i) and (ii) are duals of one another, hence we only need to prove (i)

$I_{\text{dom } C} \supseteq C \Rightarrow C \ll I$

$\Rightarrow C^\perp \subseteq I^\perp = I$ lemma and $I$ total and monotonic

$\Rightarrow B_C(I) = (I \land C^T) \lor C^\perp$

$= (I \lor C^\perp) \land C^T$ - $C$ monotonic

$= I \land C^T \leq I$

QED (i)

QED lemma

Lemma - identity

$B_{\emptyset} = \text{id}$

Proof - special case of the above

Lemma - absorption

$C'$ monotonic and $C \subseteq C' \Rightarrow B_C \circ B_{C'} = B_{C'} = B_C \circ B_{C'}$

Proof

By previous lemmas $C^T \supseteq C'^T$ and $C^\perp \subseteq C'^\perp$

The proof of both results then follows by juggling the order of $\land$s and $\lor$s

We will prove the first result only as the proofs are so similar.

$B_C \circ B_{C'}(I) = (( ((I \land C^T) \lor C^\perp) \land C'^T ) \lor C'^\perp$

$= ((I \land C^T) \lor C^\perp \land C'^T) \lor C'^\perp$ - $C'$ monotonic

$= ((I \land C'^T) \lor C^\perp \land C^T) \land C'^T$ - $C^\perp \subseteq C'^\perp$

$= (I \land C'^T \land C^T) \lor C'^\perp$ - $C'$ monotonic

$= (I \land C'^T) \lor C'^\perp$

$= B_{C'}(I)$

QED lemma

In particular if $C$ is monotonic we have:

$B_{C \oplus (a \rightarrow d)} \circ B_C = B_{C \oplus (a \rightarrow d)}$

If we are to use $B_C$ in fixed point constructions we also need monotonicity:
Lemma

$\mathcal{B}_C$ is monotonic for any $C$

Proof

say $I_1 \geq I_2$

let $I_1' = \mathcal{B}_C(I_1), I_2' = \mathcal{B}_C(I_2)$

$\forall a \in \text{App} \quad I_1(a) \geq I_2(a) \implies$

$I_1'(a) = (I_1(a) \land C^\top(a)) \lor C^\bot(a) \geq (I_2(a) \land C^\top(a)) \lor C^\bot(a) = I_2'(a)$

QED  lemma

Note that this proof applies to all $C$ and $I$ not just monotonic ones. However, if $C$ is not monotonic then $\mathcal{B}_C(I)$ will not be monotonic, even when $I$ is.

Lemma

$\mathcal{B}_C$ is monotonic as a function of $C$ (with the $\leq$ ordering on Config) - again trivial

Note that $\mathcal{B}_C$ is not monotonic with respect to the $\subseteq$ and $\ll$ orderings on Config.

6.4. Constrained iterator

To find the fixed point of $H$ we can iterate using it and starting at $\bot$. Pending analysis uses a constrained iterator, and we obtain this using the bounding function.

Definition

$H_C = \mathcal{B}_C \circ H$

This inherits obvious monotonicity properties:

Lemma

$H_C(I)$ is monotonic both as a function of $I$ and of $C$

$H_{\{\}} = H$

$C \leq C' \implies H_C \leq H_{C'}$

$C \subseteq C' \implies \mathcal{B}_C \circ H_C = H_{C'} = \mathcal{B}_{C'} \circ H_C$

Proof

all follow straight-forwardly from the corresponding properties of $\mathcal{B}_C$

QED  lemma

The second two of these properties are reflected in their fixed points:

$\text{fix } H_{\{\}} = \text{fix } H$

$C \leq C' \implies \text{fix } H_C \leq \text{fix } H_{C'}$

It seems obvious that if we add an extra assumption to a constrained iterator that is bigger or smaller than the actual value at the fixed point, then the resulting fixed point of the new iterator will be likewise too big or small. The following lemmas prove this, first by proving a similar property for single applications, and then for the fixed points. Because it is the minimal fixed points we are interested in the proofs are not duals, and have a very different form.
Lemma

(i) \((H_C(I))_a \geq d\) \(\Rightarrow\) \(H_C \oplus (a \rightarrow d)(I) \leq H_C(I)\)
(ii) \((H_C(I))_a \leq d\) \(\Rightarrow\) \(H_C \oplus (a \rightarrow d)(I) \geq H_C(I)\)
(iii) \((H_C(I))_a = d\) \(\Rightarrow\) \(H_C \oplus (a \rightarrow d)(I) = H_C(I)\)

Proof

(i) + (ii) \(\Rightarrow\) (iii)
Also (i) and (ii) are duals, so we need only prove (i)
let \(d' = (H(I))_a, d'' = (H_C(I))_a\)
then \(d'' = (d' \wedge C^T(a)) \vee C^L(a)\)
\(\Rightarrow\) \(d'' = (d' \wedge C^T(a)) \vee C^L(a)\)
so \(d \leq d''\) \(\Rightarrow\) \(C \oplus (a \rightarrow d)_a \leq d''\) - monotonicity of \(\vee\) and \(\wedge\)
Also \((H_C I)_{\text{dom } C} = C\)
\(\Rightarrow\) \((H_C I)_{\text{dom } C \oplus (a \rightarrow d)} \geq C \oplus (a \rightarrow d)\)
\(\Rightarrow\) \(B_{C \oplus (a \rightarrow d)}(H_C I) \leq H_C I\) - fixed points of \(B_{C \oplus (a \rightarrow d)}\)
That is \(H_C \oplus (a \rightarrow d)(I) \leq H_C(I)\) - absorption
QED

Lemma

\((\text{fix } H_C)_a \geq d\) \(\Rightarrow\) \(\text{fix } H_C \oplus (a \rightarrow d) \leq \text{fix } H_C\)

Proof

let \(I = \text{fix } H_C\)
then
\((H_C(I))_a \geq d\)
\(\Rightarrow\) \(H_C \oplus (a \rightarrow d)(I) \leq H_C(I) = I\) - above lemma
\(\Rightarrow\) \(\text{fix } H_C \oplus (a \rightarrow d) \leq I = \text{fix } H_C\)
QED

Lemma

\((\text{fix } H_C)_a \leq d\) \(\Rightarrow\) \(\text{fix } H_C \oplus (a \rightarrow d) \geq \text{fix } H_C\)

Proof

Not a dual of the above!! - because \(\text{fix}\) is minimal fixed point.
Consider any \(J\) such that -
\(J \leq \text{fix } H_C \oplus (a \rightarrow d)\) and \(J \leq \text{fix } H_C\)
we want to prove that -
\(H_C(J) \leq H_C \oplus (a \rightarrow d)(J)\)
now \(J \leq \text{fix } H_C \Rightarrow (H_C J)_a \leq d\)
\(\Rightarrow\) \(H_C \oplus (a \rightarrow d) J \geq H_C J\) - above lemma
QED

Lemma
Lemma

\[(\text{fix } H_C)_a = d \Rightarrow \text{fix } H_C @ (a \rightarrow d)(I) = \text{fix } H_C(I)\]

Proof

By applying the above two lemmas.

QED lemma

6.5. Incremental fixed point theorem

Theorem - incremental fixed point theorem

let \(g(d) = (H_C(\text{fix } H_C @ (a \rightarrow d))_a\)

and \(d_{fix} = \text{fix } g\)

and \(C_{fix} = C @ (a \rightarrow d_{fix})\)

then

(i) \((\text{fix } H_C)_a = d_{fix}\)

(ii) \(\text{fix } H_C = \text{fix } H_{C_{fix}}\)

Proof

First note that (ii) follows from (i) and the above lemma, therefore we only strictly need to prove (i), in fact we proceed with a little of each.

Steps -

(1) show that \(d \leq d_{lhs} \Rightarrow g(d) \leq d_{lhs}\)

where \(d_{lhs} = (\text{fix } H_C)_a\)

Proof step 1

\[d \leq d_{lhs} \Rightarrow \text{fix } H_C @ (a \rightarrow d) \leq \text{fix } H_C\]

- by above lemma

\[\Rightarrow H_C \circ \text{fix } H_C @ (a \rightarrow d) \leq H_C \circ \text{fix } H_C = \text{fix } H_C\]

- monotonicity of \(H_C\)

\[\Rightarrow g(d) = (H_C \circ \text{fix } H_C @ (a \rightarrow d))_a \leq (\text{fix } H_C)_a\]

QED (1)

(2) \(\Rightarrow d_{fix} = \text{fix } g \leq d_{lhs}\)

(3) \(\Rightarrow \text{fix } H_{C_{fix}} \leq \text{fix } H_C\)

- by above lemmas

(4) let \(I = \text{fix } H_{C_{fix}}\)

prove that \(H_C I = I\)

(4.1) show \((H_C I)_a = d_{fix}\)

- immediate from def’n of \(g\) and \(g(d_{fix}) = d_{fix}\)

(4.2) \(\Rightarrow H_{C_{fix}} I = H_C I\)

- from previous lemma

(4.3) That is \(I = (H_C I)\)

QED (4)

(5) thus \(\text{fix } H_{C_{fix}} = I \geq \text{fix } H_C\) - minimality of \(\text{fix } H_C\)

QED (ii)

(6) \(\text{fix } H_C = H_C(\text{fix } H_C) = H_C(\text{fix } H_{C_{fix}})\)

\[\Rightarrow (\text{fix } H_C)_a = g(d_{fix}) = d_{fix}\]

QED (i)

QED theorem
6.6. The algorithm

The algorithm is virtually identical to that for \( X_C \):

\[
X_C : \text{Appl} \rightarrow \mathcal{D}
\]

\[
X_C|_a = C|_a \quad a \in \text{dom } C
\]

\[
X_C|_a = \text{fix } h \quad \text{otherwise}
\]

where

\[
h(d) = Bnd_C(a)(H X_C|_{a \to d})_a
\]

fix is used here to mean the computationally effective function -

\[
\text{fix} : (\mathcal{D} \rightarrow \mathcal{D}) \rightarrow \mathcal{D}
\]

\[
\text{fix}(g) \rightarrow \text{fixit}_g(\bot)
\]

\[
\text{fixit}_g(d) \rightarrow
\]

\[
\text{let } d_{\text{new}} = g(d)
\]

\[
\text{if } d_{\text{new}} = d
\]

\[
\text{then } d
\]

\[
\text{else } \text{fixit}_g(d_{\text{new}})
\]

If the domain is finite and \( g \) always terminates then \( \text{fix}(g) \) terminates. Further if \( g \) is monotonic then this function yields the minimum fixed point of \( g \), and the formula for \( d_{\text{new}} \) can be simplified to \( g(d) \). Note also that if \( \mathcal{D} \) is a binary domain then the first application of \( g \) yields the fixed point. That is -

\[
\mathcal{D} \text{ binary } \Rightarrow \text{fix } g \Rightarrow g(\bot)
\]

With this observation the algorithm collapses to simple pending analysis.

\( Bnd_C \) is the one point equivalent of \( B_C \), that is -

\[
Bnd_C(a)(d) = (d \wedge C^=(a)) \vee C^4(a)
\]

It is used to emphasise the locality of the calculation (it only involves \( C \) and \( a \)).

We now prove termination and correctness of \( X_C \).

6.7. Termination

Theorem

\( X_C \) terminates

Proof

Induction on size of \( C \)

We assume that \( X_C \) terminates for any \( C \) when \( \|C\| > C \)

Base case \( C \) total

In this case \( X_C = C \) everywhere, and thus terminates.

QED base case
Inductive case, prove \( \mathcal{X}_\mathcal{C}_a \) terminates for all \( a \),
there are two cases

\begin{itemize}
  \item either \( a \in \text{dom} \mathcal{C} \)
    \begin{itemize}
      \item whence \( \mathcal{X}_\mathcal{C}(a) \rightarrow \mathcal{C}_a \) and terminates
    \end{itemize}
  \item or \( \mathcal{X}_\mathcal{C}(a) \rightarrow \text{fix } h \)
    \begin{itemize}
      \item where \( h(d) = \text{bdn}_\mathcal{C}(a) \text{ H } \mathcal{X}_\mathcal{C} ( a \rightarrow d ) \)
    \end{itemize}
\end{itemize}

now \( \| \mathcal{C} \oplus (a \rightarrow d) \| \geq \| \mathcal{C} \| \)

so \( \mathcal{X}_\mathcal{C} ( a \rightarrow d ) \) terminates by induction, \( \text{H} \) terminates as given,
also \( \text{bdn}_\mathcal{C}(a) \) terminates as it has only to search a finite
configuration and perform \( \lor \)s and \( \land \)s.
thus \( h \) terminates.

and \( \text{fix} \) always terminates on finite domains.
(Strictly need to prove \( h \) monotonic, this falls out of the correctness proof)
Thus \( \mathcal{X}_\mathcal{C}(a) \) terminates.

QED inductive cases

6.8. Correctness

By correctness, we mean that \( \mathcal{X}_\mathcal{C} \) yields the same result as \( \text{fix } \mathcal{H}_\mathcal{C} \), and in particular for any application of
interest \( \mathcal{X}_{\mathcal{O}}(a) = (\text{fix } \mathcal{H})_a \). The proof is nearly identical to previous proofs.

Theorem
\[ \mathcal{X}_\mathcal{C} = \text{fix } \mathcal{H}_\mathcal{C} \]

Proof

Induction on size of \( \mathcal{C} \)
We assume that \( \mathcal{X}_\mathcal{C} = \text{fix } \mathcal{H}_\mathcal{C} \) for any \( \mathcal{C} \) when \( \| \mathcal{C} \| > \mathcal{C} \)

Base case \( \mathcal{C} \) total
In this case \( \mathcal{H}_\mathcal{C} \) is constantly \( \mathcal{C} \) and thus \( \text{fix } \mathcal{H}_\mathcal{C} = \mathcal{C} \)
Similarly \( \mathcal{X}_\mathcal{C} \) is constantly \( \mathcal{C} \)

QED base case
Induction

We want to prove \( X_C(a) = (\text{fix } H_C)_a \)

there are two cases:

either \( a \in \text{dom } C \)

\[ X_C(a) \rightarrow C_a \]

but \( \text{fix } H_C \downarrow \text{dom } C = C \)

\( \text{QED } a \in \text{dom } C \)

or \( a \notin \text{dom } C \)

\[ X_C(a) \rightarrow \text{fix } h \]

where \( h(d) = \text{Bnd}_C(a) \circ (H X_C \oplus (a \to d)) \)

now \( \| C \oplus (a \to d) \| \geq \| C \| \)

\[ \Rightarrow H X_C \oplus (a \to d) = H \text{fix } H_C \oplus (a \to d) \]

and \( \text{Bnd}_C(a) = (H(\text{fix } H_C \oplus (a \to d)))_a = (\text{B}_C(H(\text{fix } H_C \oplus (a \to d)))_a \]

That is \( h \) here is equal to \( g \) of the incremental fixed point theorem,

and as \( g \) is monotonic the \( \text{fix } h \) is the minimal fixed point of \( g \).

We thus apply part (i) of the theorem :-

\[ X_C(a) = \text{fix } g = (\text{fix } H_C)_a \]

\( \text{QED } a \notin \text{dom } C \)

\( \text{QED } \) inductive cases

7. An efficient algorithm for configuration searching

The speed of the above algorithm depends critically on the speed of the \( \text{Bnd} \) function and (equivalently) augmentation \( C \oplus (a \to d) \). A simple linear search of the configuration for all elements greater than or less than a given element will be too slow for large configurations. The simple limiting function \( L \) is easy to implement as this involves only looking for an exact match, it is the finding of related elements that is of difficulty. An algorithm is proposed that uses the lattice structure of the parameter domain as a means of structuring the configuration storage. Searching for elements above or below a certain element then involves a walk through the lattice starting at the top or bottom. If the lattice were complete, corresponding to a complete configuration, we could choose any path when there were several elements less than the target (assuming a walk down) as they would all eventually lead to the same target. If the configuration is not complete then we need to explore all paths (although some would later meet) in order to find all elements above the one of interest.

We can clearly index the configuration directly by the particular function, hence we can assume we are dealing with a configuration that only represents one function, and the \( \text{App} \) domain is identical to the argument domain \( D_a \).

\[ C = D_a 
\]

The argument and algorithm is in no way dependent on this assumption as the application domain \( \text{App} \) has a perfectly good lattice structure of its own, however this would probably be a less efficient use of the algorithm. For other specific domains there may be similar specialisations that will improve performance.

We will look for a structure (call it \( \text{Env} \)) that represents the configuration. We will need several operations from this structure.

(i) \( \text{Find } C^\lor = \bigvee \limits_{a \geq a} C(\ a' ) \)

(ii) \( \text{Find } C^\land = \bigwedge \limits_{a \leq a} C(\ a' ) \)

(iii) \( \text{Find } C \oplus (a \to d) = (d \land C^\lor(\ a' )) \lor C^\land(\ a ) \)

That is we will want functions:
\[ upper\_bound \quad : \quad Env \times D_a \rightarrow D_d \]
\[ lower\_bound \quad : \quad Env \times D_a \rightarrow D_d \]
\[ augment \quad : \quad Env \times D_a \times D_d \rightarrow Env \]

We will want these to mirror the actual operations, that is, if \( env \) represents \( C \) then:

\[ upper\_bound(\ env, a ) = C^\triangledown( a ) \]
\[ lower\_bound(\ env, a ) = C^\triangledown( a ) \]
\[ augment(\ env, a, d ) \] represents \( C \oplus ( a \rightarrow d ) \)

\( Env \) will both represent the configuration itself, but also it will contain information specifically to improve the speed of the searches implied by the \( \land \)s and \( \lor \)s.

We represent the configuration value by a set of nodes, each one corresponding to an argument in the domain of \( C \).

\[
Env \equiv \text{nodes} : \text{IP} \text{ Node} \\
\quad \quad \quad \quad \quad \text{arg} : \text{Node} \rightarrow D_a \\
\quad \quad \quad \quad \quad \text{val} : \text{Node} \rightarrow D_d
\]

where
- \( \text{dom arg} = \text{dom val} = \text{nodes} \)
- \( \text{range arg} = \text{dom } C \)
- \( \text{arg} \) is injective
- \( \forall n \in \text{nodes} \ \text{val}( n ) = C( \text{arg}( n ) ) \)

In order to search this structure, we will need to cross index these nodes giving for each node the set of nodes with arguments "just above" and "just below". This will give the nodes a structure like the lattice \( \text{dom } C \).

\[
Env +\equiv \text{up\_nodes} : \text{Node} \rightarrow \text{IP( Node )} \\
\quad \quad \quad \quad \quad \text{down\_nodes} : \text{Node} \rightarrow \text{IP( Node )}
\]

where
- \( \text{dom up\_nodes} = \text{dom down\_nodes} = \text{nodes} \)
- \( \forall n \in \text{nodes} \ \text{up\_nodes}( n ) \subseteq \text{nodes} \)
- \( \forall n, n' \in \text{nodes} \ \text{not } \exists n'' \in \text{nodes} \ \text{st } \text{arg}( n ) < \text{arg}( n'' ) < \text{arg}( n' ) \)
- \( \forall n, n' \in \text{nodes} \ \text{not } \exists n'' \in \text{nodes} \ \text{st } \text{arg}( n' n' ) < \text{arg}( n'' ) < \text{arg}( n' ) \)

We will need to record the topmost and bottommost nodes in order to start searches. We can do this explicitly:

\[
Env +\equiv \text{top\_nodes} : \text{IP( Node )} \\
\quad \quad \quad \quad \quad \text{bottom\_nodes} : \text{IP( Node )}
\]

where
- \( \text{top\_nodes} = \{ \text{node } \in \text{nodes} \mid \text{up\_nodes}( \text{node } ) = \{ \} \} \)
- \( \text{bottom\_nodes} = \{ \text{node } \in \text{nodes} \mid \text{down\_nodes}( \text{node } ) = \{ \} \} \)

However this will lead to special cases when we are dealing with elements at the top or bottom of \( Env \). An alternative is to add special elements to \( D_a \) and \( D_d \), a "super top" (\( \top \)) and "super bottom" (\( \bot \)). These will sit above and below the normal top and bottom and thus any real argument will sit below the top node and above the bottom. So instead of the sets \( \text{top\_nodes} \) and \( \text{bottom\_nodes} \) we will have:
Env  +=  top_node : Node
        bottom_node : Node

where
    top_node, bottom_node ∈ nodes
    arg( top_node ) = ⊤_a
    arg( bottom_node ) = ⊥_a
    val( top_node ) = ⊤_d
    val( bottom_node ) = ⊥_d

All three operations we wish to carry out depend on first finding the nodes directly above and below the target argument. That is we want functions:

above  : Env × D_a → IP( node )
below  : Env × D_a → IP( node )

where
    above( env, a ) ⊆ nodes
    below( env, a ) ⊆ nodes
    not ∃ n ∈ nodes, n ∈ above( env, a ) st a < arg( n ) < arg( n' )
    not ∃ n ∈ nodes, n' ∈ below( env, a ) st arg( n' ) < arg( n ) < a

Given such functions we can easily specify the operations we want in terms of them.

upper_bound  : Env × D_a → D_d
upper_bound( env, a ) = ⋀_{n ∈ above( a )} val( n )

lower_bound  : Env × D_a → D_d
lower_bound( env, a ) = ⋁_{n ∈ below( a )} val( n )

augment  : Env × D_a × D_d → Env
let A = above( a ), B = below( a )
if A = B = { n }  − that is a ∈ dom C
    augment( env, a, d ) = env
otherwise  − adding a new element
    augment( env, a, d ) = env'
where
    dom env' = dom env + { n }
    arg( n ) = a
    val( n ) = d
    up_nodes( n ) = A
    down_nodes( n ) = B
    arg( n' ) = arg( n' )  n' ≠ n
    val( n' ) = val( n' )  n' ≠ n
    up_nodes( n' ) = up_nodes( n' )  n' ≠ n  and  n' ∉ B
    up_nodes( n' ) = up_nodes( n' )  A + { n }  n' ∈ B
    down_nodes( n' ) = down_nodes( n' )  n' ≠ n  and  n' ∉ A
    down_nodes( n' ) = down_nodes( n' )  B + { n }  n' ∈ A

It only remains to define effective algorithms for above and below. In fact, we can define two algorithms for each, depending on when we start at the top or the bottom.
\[above_{\text{Env}}^{\uparrow} (a) = above_{\text{Env}}^{\uparrow} (a, \text{top\_node})\]
\[above_{\text{Env}}^{\downarrow} (a, b) = \begin{cases} \text{let } S = \bigcup \{ above_{\text{Env}}^{\uparrow} (a, c) \mid c \in \text{down\_nodes}(b) \text{ and } a \leq \arg(c) \} \\ \text{if } S = \{ \} \\ \text{then } b \\ \text{else } S \end{cases}\]
\[below_{\text{Env}}^{\uparrow} (a, b) = below_{\text{Env}}^{\uparrow} (a, \text{top\_node}) \lor \bigcup_{b \in above_{\text{Env}}^{\downarrow} (a)} below_{\text{Env}}^{\downarrow} (a, b)\]
\[below_{\text{Env}}^{\downarrow} (a, b) = \begin{cases} \text{if } \arg(b) \leq a \text{ and } \forall c \in \text{up\_nodes}(b) \text{ not } \arg(c) \leq a \\ \text{then } b \\ \text{else if } \arg(b) \leq a \\ \text{then } \{ \} \\ \text{else } \bigcup_{c \in \text{down\_nodes}(b)} below_{\text{Env}}^{\downarrow} (a, c) \end{cases}\]

\(below_{\text{Env}}^{\downarrow}\) is the dual of \(above_{\text{Env}}^{\uparrow}\), working from the bottom, and \(above_{\text{Env}}^{\downarrow}\) is the dual of \(below_{\text{Env}}^{\uparrow}\). The choice between using upward or downward searches could be made once and for all, or could be made depending on the exact value of \(a\).

The cost of an application of \(above_{\text{Env}}^{\downarrow}\) is proportional to the number of nodes with \(\arg(\text{node}) \geq a\). In the worst case (looking for \(\bot\)) this is \(\parallel \text{nodes} \parallel\) which is as bad as the simple search (but with a worse constant of proportionality). Typical cases are far better. The worst kind of domain is a simple linear one, when one visits half the nodes on average. However the argument domains will usually be product domains and here the behaviour is much better. For a product of \(n\) linear domains one would visit on average \(\frac{n}{2}\) of the nodes. Similarly the cost of \(below_{\text{Env}}^{\uparrow}\) is proportional to the number of nodes with \(\arg(\text{node}) \leq a\) with a worst case looking for \(\top\).

This suggests that by examining where the target lies we could protect ourselves against worst case behaviour, but because the exact cost depends not only on \(a\) but also \(\text{dom}\; \mathcal{C}\) it may not be worth it.

The above cost assumes that the unions can be performed in linear time implying a complex set representation (eg. hashing), but an imperative version of the algorithm would run down the paths in turn and mark nodes encountered which could then be avoided by subsequent paths.

In the incremental fixed point algorithm, the applications of \(\mathcal{B}n_{\mathcal{C}}\) and \(\mathcal{C} \oplus (a \rightarrow d)\) come in the expression \(\text{fix } h\) where \(h\) is defined as:

\[h(d) = \mathcal{B}n_{\mathcal{C}}(a) \circ (H \; X_{\mathcal{C} \oplus (a \rightarrow d)})_{a}\]

Now on each iteration of \(\text{fix}\) we need to find out \(above(a)\) and \(below(a)\) to work out \(\mathcal{C}^{T}(a)\) and \(\mathcal{C}^{\perp}(a)\) in \(\mathcal{B}n_{\mathcal{C}}(a)\) and also to calculate \(\mathcal{C} \oplus (a \rightarrow d)\). These are the same both for these three uses in each iteration, but also between iterations. Therefore these can be calculated once and then reused on each iteration. That is we only need one configuration lookup for each fixed point calculation. Further as the number of iterations of the fixed point is itself on average half the domain height, cost of configuration search is likely to be far less than the cost of iteration. If you include the reduction in the number of iterations because we start above bottom (typically very near the fixed point), the algorithm looks very respectable.

If the height if the configuration is greater than the height of the result domain, it may be worth using frontier analysis techniques to compress “flat” chains. However care must be taken to keep note of those applications that are in the configuration but which are holophrasted for efficiency. For some algorithms even this may not be necessary.

In this section we have seen how configurations can be stored and accessed efficiently with a worst case behaviour comparable with simple searching, but with typical performance far better. However, it is also noted that we can reuse most of the work so that searches are relatively infrequent in the incremental fixed point algorithm, hence even worst case behaviour is likely to be acceptable.
8. Higher order analysis using term algebras

The two weaknesses of pending analysis were the limitation on the domain, and that it is a first order analysis. We have seen that it can be modified to cope with large domains, and thus of course if necessary function domains. However, these are large and comparisons over the domains, either of equality (for $X^2$) or inequality (for $X_Y$) will be very expensive. In general, the number of distinct partial applications will be small compared to the total number of functions available. It would be nice therefore to use the terms describing the functions, instead of the functions themselves, and use these as the pending functions. For example, in the analysis of map $\mathbf{f}$ where $\mathbf{f}$ is some function, we would pend the application of map to the identifier $\mathbf{f}$. This is equivalent to doing pending analysis over a domain of terms, and as the analysis is correct for all domains it is correct in particular for term algebras.

The only misgiving one might have, is that we are distinguishing terms that may be the same. One doubt arising from this is the impact on the efficiency of the algorithm as we miss some pending calls, however this is likely to be far less than the cost of full function comparison. More problematical is that it might affect the correctness of the algorithm. We know that it is correct within its own domain (term algebras), but is it correct in the original higher order domain. In fact this is ok, as the term algebra is a reified interpretation (opposite of abstract interpretation) of the original domain and functions. That is if $\text{Term}$ is the evaluation operation from terms to values we have:

$$\forall f, x \quad \text{Term}(f) \quad \text{Term}(x) = \text{Term}(f \ x)$$

Of course the pending analysis only works if the domains are finite, the term domain is in principle infinite. Does this invalidate the procedure? In many cases it does not, as the terms generated are strictly bounded, the cases where pending analysis will produce unbounded terms are rare. They can only occur where a function can be applied to an irreducible term which includes a partial application of itself. The possibility of this can be detected by static analysis (yet more abstract interpretation) but this is not essential as dynamic measures are just as effective. When a term is generated which exceeds a certain complexity, we can substitute a "super-top" element ($\top$) that has the property that any irreducible term containing it becomes super-top too, and when it is applied to anything, or is the argument to a primitive operation, the result is larger than any other such application. That is:

$$\top(a) = \bigvee_{f \in \mathbb{T}} f(a)$$
$$f(\top) = \bigvee_{a \in \mathbb{T}} f(a)$$

Or alternatively and cheaper, but less exact:

$$\top(a) = \top$$
$$f(\top) = \top$$

With either of these we loose the exact correspondence between the normal interpretation and the reified term algebra, however both of these yield a relationship

$$\forall f, x \quad \text{Term}(f) \quad \text{Term}(x) \preceq \text{Term}(f \ x)$$

Which is sufficient to yield a safe, but inexact, result of pending analysis. This is no worse in its effect than the pessimizing suggested by Young and Hudak to reduce evaluation time, and can similarly be varied by adjusting the acceptable complexity of terms before "topping out".

We do not need to change our proofs at all, except that proofs of properties of functionals defined using terms must include an apply operator, which is in fact monotonic on these domains and does not alter their correctness.

9. Fixed point properties of configurations

Young and Hudak suggest memoing the results of pending analysis as an optimisation. The resulting algorithm no longer has the simple functional form used previously, and it needs additional proof. However the memoing does suggest that one is iterating estimates to a whole configuration, rather than just at a point. This section looks at the fixed point properties of configurations in order to establish a general method of
proving correct optimisations of basic pending analysis.

\[ \forall I \quad C \subseteq I \implies C \subseteq HI \]

Equivalently (and easier to prove!)

\[ C \subseteq HC \]

**Theorem**

if \( C \) is a fixed configuration of \( H \) then

\[ \exists I \text{ a fixed point of } H \quad \text{(not necessarily minimal)} \]

\[ \quad \text{st} \quad C \subseteq I \]

**Proof**

let \( I_0\vert_{\text{dom } C} = C \)

\[ I_0\vert_{a} = \bot \quad \text{elsewhere} \]

let \( I_{i+1} = HI_I \)

then

\[ C \subseteq I_0 \]

\[ \forall i \quad C \subseteq I_i \implies C \subseteq I_{i+1} \]

\[ \quad \text{and } \bigvee I_i \text{ is a fixed point of } H \]

QED

So if \( C \) is a fixed configuration and \( C \subseteq \text{fix } H \) then we can conclude that \( C = \text{fix } H_{\text{dom } C} \).

This is very useful for proving algorithms correct. We need only prove that any value obtained is part of a fixed configuration to know that the value is safe (i.e. above the fixed point). This is an end state result and is not dependent on the internal workings of the algorithm. To prove the algorithm optimal (that is equal to the minimum fixed point) will typically involve a process argument that the minimal fixed point is never exceeded, however since it is a weak (i.e. \( \leq \)) argument it is easier to work with than an exact argument and it is thus easier to involve clever heuristics.

If we are happy with safe results, we need only prove that

\[ \forall I \quad C \gg I \implies C \gg HI \]

or

\[ C \gg HC \]

whence by a similar proof to the above we have \( C \gg \text{fix } H \)

Fixed configurations can be related to *minimal function graphs*.\(^6\) Jones and Mycroft describe minimal function graphs in terms of a domain with two "bottom" elements: \( ! \) representing non termination and \( \bot \) representing "never called". Roughly their \( ! \) corresponds to values of \( \bot \) in the configuration, and their \( \bot \) to values which are undefined.

**10. A non-deterministic algorithm based on configurations**

The pending analysis algorithms proved have been functional, they have not involved memoising as suggested by Young and Hudak, and have sometimes been conservative (\( Bn \) \( c \) may reduce the value returned). This is because they have relied on proving that intermediate results have been equal to fixed points of constrained iterators. The various optimisations will typically yield results that are not exactly the fixed point of the constrained iterator but lie between it and the actual fixed point required. We could go through all the proofs again and prove the various optimised versions. The above result about fixed configurations would make this process far easier. It is fairly obvious that if we take all the application result pairs from the last iteration of all the various embedded fixed point calculations that they would form a fixed configuration. we would thus only need to prove that all the values obtained are below the fixed point. This would
probably require a \( \geq \) version of the incremental fixed point theorem, but with all the lemmas developed involving \( \mathcal{B} \), this would be easy.

An easy way to prove such optimisations is to prove correct a non-deterministic algorithm which includes the optimisations of interest as special sub-cases. Rather than doing this for top-down pending analysis we examine an alternative non-deterministic algorithm that is based around bottom-up evaluation of configurations.

Define a sequence

\[
\text{let } C_i \text{ be a sequence of configurations defined thus:} \\
C_0 \text{ any configuration such that } C_0 \ll \text{fix } H \\
\text{for each } i \text{ either} \\
(1) \text{ choose } a \in \text{dom } C_i, \ a \leq (\text{fix } H)_{a} \\
C_{i+1} = C_i \uparrow (a \rightarrow d) \\
\Rightarrow C_{i+1} \ll \text{fix } H \\
(2) \text{ choose } C \subseteq C_i \text{ then choose } C' \subseteq H \ C \\
C_{i+1} = C_i \uparrow C' \\
\Rightarrow C_{i+1} \ll \text{fix } H \\
(3) \text{ choose any monotonic } C_{i+1} \ll C_i \\
\Rightarrow C_{i+1} \ll \text{fix } H
\]

Obtain result

\[
\text{for any } i \text{ choose } C \subseteq C_i \text{ a fixed configuration} \\
\Rightarrow C \subseteq \text{fix } H
\]

The operator \( \uparrow \) is the symmetric upward sum, defined thus:

\[
\text{Define } - \text{dom } \ C \uparrow C \\
\text{dom} (C \uparrow C') = \text{dom } C \cup \text{dom } C' \\
(C \uparrow C')_{a} = \bar{C}_{a} \lor \bar{C}'_{a}
\]

The various algorithms used can all be seen as variants of this. Consider, for example, general pending analysis with the bounding function. It uses the \( \oplus \) operator, however this can be described as step 1 or 2 followed by step 3. To avoid saying this we will describe the variant of the algorithm using the \( \uparrow \) operator.

The algorithm starts off with an empty configuration. Each time a new value is encountered step 1 is executed with \( d = \perp \) then the fixed point calculation begins. Effectively doing multiples of step 2. As the functional algorithm does not remember internal results of recursive calls, step 3 is effectively executed at each function return. By simply omitting the “forgetful” step 3, we produce pending analysis with memoing.

However, we can imagine far cleverer heuristics based on this algorithm. In general pending analysis can spend effort calculating values that do not contribute to the final result (but are needed for intermediate values). If instead of pushing execution right down the call tree, we could as a first estimate return bottom, for everything, then (if the pending value has been used) iterate on this value alone, only when this has stopped do we go round the other values used to make sure they are correct. This means that some applications low in the domain may not need to be iterated over. We can retain dependency information in order to decide which values may have changed, and also remember critical paths, so that we can first iterate on applications likely to lead to an increase in the target application.

There are clearly many ways of approaching this, but typically algorithms will have one or more target applications that they’re actually after. If the call graph of the application involves elements not in the configuration, step 1 will be executed to extend it. Then by examining the call graph an application is chosen as “pivot” and step 2 is iterated until stable or the result of the pivot increases enough to change one of the targets. In order to execute step 2 of course the configuration is typically further extended etc.
Following these heuristics the fixed configuration output will be precisely the minimal function graph associated with the target applications. Other values will have been calculated of course, as it is impossible in general to calculate the minimal function graph without doing some "wasted" work.

A memoised algorithm, that is one without step 3 can be shown to terminate so long as one is reasonably sensible about choices of $C'$. Algorithms using step 3 need separate proofs, but why bother? Possibly, in some cases we could prove that certain applications will never be needed again and its worth reducing the size of configuration, but its hard to believe that this will frequently be worthwhile.

By defining a non-deterministic algorithm we have left room for many different optimisations to basic pending analysis, some of which may have improved typical behaviour. Strangely enough, we end up with an algorithm far closer to a traditional fixed point calculation.

11. Conclusions

Proofs have been given of Young and Hudak’s pending analysis for non-self applicatory functions and of the improved algorithm for general and higher order functions. In addition a new non-deterministic algorithm has been presented and proved correct. This algorithm not only includes both pending analysis and standard algorithms as special cases, but also lays the formal groundwork for more advanced algorithms. This unification of approaches is especially encouraging as it gives a framework for future study.

An open problem is whether simple pending analysis is in fact correct for general functions. Examples, such as in Appendix 7, make one very doubtful, however pending analysis does still work for this. It would be nice to either form a true counter example or find a proof however the examples here show that such a proof would be tortuous. My gut feeling is that perhaps it is correct in the binary case, but it is clear that gut feelings in this area can be very misleading.

Appendix 1 - Recursion equations for multiple functions

We have called the domain of the function sought $Appl$ as it is expected that it typically represents the application of a function symbol to some arguments. Below is briefly outlined the formal basis of this. None of the proofs depended on the well typedness of the operations involved, and if the relevant definitions are initially well typed then the result of the various algorithms will themselves be well typed.

The abstract interpretation will be over some set of recursively defined functions mapping into finite lattices. The functions form a signature $<F, T, arity>$:

$\begin{align*}
F & : \text{a set of function symbols} \\
T & : \text{a set of type symbols} \\
arity & : F \rightarrow T^* \times T
\end{align*}$

The domains of interest are labelled by these types, and for convenience we consider also the disjoint sum of these domains.

$\mathcal{D} = \sum_{\tau \in T} \mathcal{D}_\tau$

The set of all type correct applications of function symbols to members of $\mathcal{D}^*$ we will call $Appl$:

$\begin{align*}
Appl = \{ \text{fd}_1d_2..d_n \mid f \in F \text{ and } d_i \in \mathcal{D}_\tau \}
\end{align*}$

where

$arity(f) = (\tau_1, \tau_2 .. \tau_n)$,

This inherits a partial ordering from the domains:

$fd_1d_2..d_n \leq f'd_1'd_2'..d_n' \equiv f = f' \text{ and } d_1 \leq d_1' \text{ and } d_2 \leq d_2' .. \text{ and } d_n \leq d_n'$

Any interpretation is a $\Sigma$-algebra over this signature. And the set of all such interpretations we call $\text{Interp}$. 
\[ \text{Interp} = \text{Appl} \rightarrow \mathcal{D} \]
subject of course to the well typing condition:
\[ \forall a = f d_1 d_2 \ldots d_n \in \text{Appl}, \ I \in \text{Interp} \]
\[ I( a ) \in \mathcal{D}, \]
where \( \text{arity}( f ) = ( \tau_1, \tau_2, \ldots, \tau_n ) \).

The recursion equations can be represented as a function on interpretations:
\[ G : \text{Interp} \rightarrow \text{Interp} \]

These equations are of course usually represented by terms over the function symbols, but we do not need this additional structure for the first part of the analysis. The only properties we require is that \( G \) is monotonic and continuous. (We will not need to invoke continuity in our arguments explicitly as all the domains of interest are finite, and it will thus be implicitly true of all monotonic functions.)

The problem we are after solving is therefore finding \( \text{fix } G \), the minimal fixed point of \( G \).

**Appendix 2 - Facts used about fixed points**

In this paper we have used the fixed point operator (called \( \text{fix} \)) extensively. Its properties are well known, and those used in the paper are summarise here.

The minimal fixed point of a function \( f \) is the smallest value \( x \) such that \( f( x ) = x \). Thus we have the important conclusion that:
\[ f( x ) = x \Rightarrow \text{fix } f \leq x \]

A standard result is that the minimal fixed point of a function \( f \) is given by:
\[ \text{fix } f = \bigvee x_i \]
where
\[ x_0 = \perp \]
\[ x_{i+1} = f( x_i ) \]

This depends on the function \( f \) being monotonic and continuous however, as the lattices we will consider will always be finite the continuity condition need not concern us.

Two facts about \( \text{fix} \) that result fairly immediately from the above are:

(i) \( x \leq \text{fix } f \Rightarrow f( x ) \leq \text{fix } f \)
(ii) \( ( x \leq y \Rightarrow f( x ) \leq y ) \Rightarrow \text{fix } f \leq y \)
(iii) \( ( x \leq \text{fix } f \text{ and } x \leq \text{fix } g \Rightarrow f( x ) \leq g( x ) ) \Rightarrow \text{fix } f \leq \text{fix } g \)
(iv) \( x \geq f( x ) \Rightarrow x \geq \text{fix } f \)

**Appendix 3 - Proof that non-self applicatory functionals defined by terms are fully monotonic**

We want to prove that if a functional \( G \) is defined by terms and is not self-applicatory, then it is monotonic. So we need to prove:
\[ I \leq I' \Rightarrow G I \leq G I' \]

Now if \( G \) is defined by non-self applicatory terms, for any \( a \) we have
\[ G I_{a_1} = \text{Eval}_I H_a^1 \]

Where \( H_a^1 \) is a non-self applicatory term. So it suffices to prove the more general theorem below:
Theorem

\[ I \leq I' \text{ and } \text{term} \text{ is non-self applicatory} \]
\[ \Rightarrow \quad \text{Eval}_I(\text{term}) \leq \text{Eval}_{I'}(\text{term}) \]

Proof

By induction on \( \text{depth}(\text{term}) \)

Base case \( \text{depth}(\text{term}) = 0 \)
\[ \text{term} = d \]
\[ \text{Eval}_I(\text{term}) = d = \text{Eval}_{I'}(\text{term}) \]
\(QED\) base case

Inductive case \( \text{depth}(\text{term}) > 0 \)

there are two subcases

either \( \text{term} = op\ e_1 \ldots e_n \)
where the \( e_i \) are themselves non-self applicatory

\[ \text{depth}(e_i) < \text{depth}(\text{term}) \]
\[ \Rightarrow \quad v_i = \text{Eval}_I(e_i) \leq \text{Eval}_{I'}(e_i) = v_i' \]
\[ \Rightarrow \quad \text{Eval}_I(\text{term}) = op\{v_i\} \leq op\{v_i'\} = \text{Eval}_{I'}(\text{term}) \]
\(QED\) \(op\ e_1 \ldots e_n\)

or \( \text{term} = f\ e_1 \ldots e_n \)

where the \( e_i \) are constant terms

\[ \Rightarrow \quad v_i = \text{Eval}_I(e_i) = \text{Eval}_{I'}(e_i) \]
\[ \Rightarrow \quad I(\{v_i\}) \leq I(\{v_i'\}) \]
\(QED\) \(f\ e_1 \ldots e_n\)

\(QED\) inductive case

QED

Appendix 4 - Proof that all functionals defined by terms are \( \succ \) monotonic and pseudo-monotonic

To prove pseudo-monotonicity, we first prove a stronger result, namely monotonicity with respect to a new relation \( \succ \). The relation \( \succ \) is stronger than normal function comparison, but equivalent to it when either of its arguments are monotonic.

Definition \( \succ \)

\[ f \succ f' \equiv \forall a, a' \quad a \geq a' \Rightarrow f(a) \geq f'(a') \]

Note how this differs from normal \( \succeq \) relation which only covers \( a = a' \). Similarly note how if either of \( f \) or \( f' \) is monotonic, we can introduce an intermediate "pivot" term \( f(a') \) or \( f'(a) \) respectively to prove \( \succ \) from \( \succeq \).

\( \succ \) is not a true partial order in itself, as it is transitive, but not reflexive. The case where \( f \succeq f \) is of interest being exactly when \( f \) is monotonic. If we require a true partial order we can always add an "or equal" condition to \( \succ \).

\[ f \succeq f' \equiv f \succ f' \text{ or } f = f' \]

The lattice formed by \( \succeq \) has a rather complicated least upper bound operator:

\[ a \lor \succeq b = \text{if } a = b \text{ then } a \text{ else } \text{ceil}(a) \lor \text{ceil}(b) \]

The ceiling function means that it is non-local. The fixed point operator is better behaved, and if \( f \) is \( \succ \) (or \( \succeq \)) monotonic and the domain is finite then the standard fixed point operator can be used. Consider the sequence

\[ x_0 = \bot, \quad x_{i+1} = f\ x_i \]

This is \( \succ \) (or \( \succeq \)) increasing and hence will terminate at the minimal fixed point of \( f \). In the case of \( \succ \)
monotonicity, this means that this fixed point will be a monotonic function, even if the intermediate \( x_i \) are not.

As functionals defined by terms are \( \succ \) monotonic, I had at first hoped that this would give an easy proof of simple pending analysis for general functions, unfortunately the limiting functional \( \mathcal{L}_c \) is not \( \succ \) monotonic, and changes to make it so would mean something of at least the complexity of \( \mathcal{B}_c \) however, it does seem an interesting ordering waiting for an application.

We prove \( \succ \) monotonicity first and then use this to prove pseudo-monotonicity as a special case.

\( \succ \) monotonicity

We want to prove that if a functional \( G \) is defined by terms it is \( \succ \) monotonic, that is:

\[
I \succ I' \implies GI \succ GI'
\]

Now if \( G \) is defined by terms, for any \( a \) we have

\[
G I | a = Eval_I H^i_a
\]

So it suffices to prove the more general theorem below:

**Theorem**

\[
I \succ I' \implies \forall \text{ term } \geq \text{ term}' \implies Eval_I(\text{ term }) \geq Eval_I(\text{ term}')
\]

**Proof**

By induction on \( \text{ depth( term )} \)

Base case \( \text{ depth( term )} = 0 \)

\[
term = d, \ term' = d' \implies Eval_I(\text{ term }) = d \geq d' = Eval_I(\text{ term}')
\]

QED base case

Inductive case \( \text{ depth( term )} > 0 \)

there are two subcases

- either \( term = op e_1 \ldots e_n, \ term' = op e'_1 \ldots e'_n, \ e_i \geq e'_i \)

\[
\text{ depth( } e_i \text{ )} < \text{ depth( term )} \implies v_i = Eval_I(\text{ e}_i) \geq Eval_I(\text{ e}'_i) = v'_i
\]

\[
\implies Eval_I(\text{ term }) = op \{ v_i \} \geq op \{ v'_i \} = Eval_I(\text{ term}')
\]

QED

or \( term = f e_1 \ldots e_n, \ term' = f e'_1 \ldots e'_n, \ e_i \geq e'_i \)

\[
v_i \geq v'_i \text{ as above } \implies I(\{ v_i \}) \geq I(\{ v'_i \}) - \text{ as } I \succ I'
\]

QED

Pseudo-monotonicity

We want to prove that if a functional \( G \) is defined by terms it is pseudo-monotonic, that is:

\[
I \succeq I' \text{ and either } I \text{ or } I' \text{ monotonic } \implies GI \succeq GI'
\]

But if \( I \succeq I' \) and either \( I \text{ or } I' \) is monotonic then \( I \succ I' \) and thus \( GI \succ GI' \) which implies (and is equivalent in this case to) \( GI \succeq GI' \).

**Appendix 5 - Properties of \( \mathcal{L}_c \)**

We define \( \mathcal{L}_c \) for all configurations by:

**Definition** \( \mathcal{L}_c \)
\[ \text{dom} \left( \lambda_c \right) = \text{dom} \ c \]

\[ \lambda_c \ c \ | a = c_a \quad \text{if} \quad a \in \ c \]

\[ \lambda_c \ c \ 
\]

Lemma

\( \lambda_c \) is fully monotonic \((\ll)\) as a function \( \text{Config} \rightarrow \text{Config} \)

Proof

\[ \lambda_1 \ll \lambda_2 \]

let \( \lambda_1' = \lambda_c \lambda_1 \), \( \lambda_2' = \lambda_c \lambda_2 \)

\[ \text{dom} \lambda_1' = \text{dom} \lambda_1 \subseteq \text{dom} \lambda_2 = \text{dom} \lambda_2' \]

\[ a \in \text{dom} \ c \implies \lambda_1 a = \lambda_1 a = \lambda_2 a \]

\[ a \in \text{dom} \ \lambda_1 - \text{dom} \ c \implies \lambda_1 a \subseteq \lambda_2 a = \lambda_2 a \]

QED

Lemma

\( \lambda_c \) is continuous \((\ll)\)

Proof

Consider \( \{ \lambda_k \}_{k \in \Lambda} \), \( \lambda_\Lambda = \bigvee \lambda_k \)

\( \text{dom} \lambda_\Lambda = \bigcup \lambda_k \)

\( \lambda_\Lambda a = \bigcup \{ \lambda_k a \} \quad \text{if} \quad a \in \text{dom} \lambda_k \)

let \( \lambda' = \lambda_c \lambda_\Lambda \)

\( \text{dom} \lambda' = \text{dom} \lambda_c \lambda_\Lambda = \bigcup \text{dom} \lambda_\Lambda = \bigcup \text{dom} \lambda' \)

\[ a \in \text{dom} \ c \implies \lambda_\Lambda a = \lambda_a = \bigcup \lambda'_a \]

\[ a \in \text{dom} \ \lambda_\Lambda - \text{dom} \ c \implies \lambda_\Lambda a = \lambda_\Lambda a = \bigcup \lambda_\Lambda a = \bigcup \lambda_\Lambda a \]

QED

Appendix 6 - Functions defined using self partial application

Higher order pending analysis using terms had to be inexact where there was no bound to the complexity of terms produced. This can occur quite easily (and usefully) in weakly typed functions, for instance \( \text{map}_n \) that takes a function to be applied to a list of lists of depth \( n \) and returns a homomorphic list of lists.

\[ \text{map}_n\ f\ 0\ l = f\ l \]

\[ \text{map}_n\ f\ n\ l = \text{map}_n\ \text{map}\ f\ n-1\ l \]

\( \text{map} \) is the normal function applying \( f \) to each element of a list. An application of \( \text{map}_n\ f\ n\ l \) will lead to a term of the form

\[ \text{map}_n\ \text{map}\ \text{map}\ \ldots\ \text{map}\ f\ \ldots\ \text{map}\ f\ 0\ l \]

Where the depth of nesting of maps is \( n \), before it is finally reduced. Without limiting the depth of terms acceptable, pending analysis would not terminate for this example.

\( \text{map}_n \) is of course not Milner typeable, there are however type correct functions which exhibit similar properties. Let \( \text{twice} \) be a function, that takes a function and an argument, and applies the function twice

\[ \text{twice} : \ ( \ * \rightarrow \ *) \rightarrow \ * \rightarrow \ * \]

\[ \text{twice} \ f\ x = f\ (f\ x) \]

We then use this to define a function \( \text{lots} \) that applies a function to an argument lots of times
lots : nat -> ( * -> * ) -> * -> *
lots 0 f x = f x
lots n f x = lots n-1 ( twice f ) x

The application lots n f x will lead to a term:

lots 0 ( twice ( twice .. ( twice f ) .. )) x

Where there are n applications of twice (leading to $2^n$ applications of f!), before it eventually reduces. Again pending analysis would fail without a size limit on the term algebra.

Typically functions defined in this manner are rather obtuse. Consider two more such functions:

lots_more : nat -> ( * -> * ) -> * -> *
lots_more 0 f x = f x
lots_more n f x = lots_more n-1 ( lots_more n-1 f ) x

strange : ( nat -> nat ) -> nat -> nat
strange f x 0 = f x
strange f x y = strange f x y-1 y odd
strange f x y = strange (strange f y-1) x y/2 y even

Appendix 7 - a nasty non-self applicatory binary function

We have not proved that simple pending analysis works for simple binary functions works, but here is an example where a "reasonable" intermediate result fails. If pending analysis is applied to this function, it does however work!

The reasonable result is:

$$G\ G_a^{+} (a) \ fix\ G|_a \ 
\leq \ ( \ G\ G_{a} \ fix\ G )|_{a}$$

Consider the function:

$$f(x, z) = z \lor f(f(x, z), T)$$

Clearly if $G$ is the generator $fix\ G = T$, however we find that

$$( G\ G_{a^{+} a} \ fix\ G )|_{a} \ 
\geq \ ( G\ G_{a} \ fix\ G )|_{a}$$

where

$$a = T, \bot$$
$$a' = T, T$$

This is the opposite of our intuition that fixing some values to $\bot$ reduces the final result. The reason for this is of course the self applicatory expression $f(f(x, z), T)$, on the left hand side this leads to an evaluation of $f(a')$, where $a' = \bot, T$, which is not pending, hence reduces to $T$, this value then bubbles up to give a final answer of $T$. On the right however the recursive call is to $f(a')$, which leads to a result of $\bot$.

The exact call chain is below, the function applications are labelled with their level numbers and primed on the right, so that $f_1$ is an outer most call on the right, whereas $f_3$ is an innermost $(fix\ G)$ call on the left.

$$LHS = f_1(a) \rightarrow \bot \lor f_2(f_2(a), T)$$
$$\rightarrow \bot \lor f_2(\bot, T) \hspace{1cm} - a \ pending$$
$$\rightarrow \bot \lor f_3(\bot, T) \hspace{1cm} - a' \ not\ pending$$
$$\rightarrow \bot \lor T \rightarrow T$$
RHS = \( f_1' (a) \rightarrow \bot \lor f_2' (f_2' (a), \top) \)
\( \rightarrow \bot \lor f_3' (f_3' (a), \top) \) - \( a \) not pending
\( \rightarrow \bot \lor f_2' (\top, \top) \)
\( \rightarrow \bot \lor \bot \) - \( a' \) pending
\( \rightarrow \bot \)

References


Finding fixed points in non-trivial domains:
Addendum to York Report 107

Alan Dix

Full Reference
http://alandix.com/academic/papers/fixpts-YCS107-88/.NH

1 Introduction
In York Report 107, I proved the correctness of pending analysis for non-self-applicatory functions, over both binary and general finite (height) domains. I also proved slight variants of basic pending analysis, that make better use of the monotonicity of functions correct for all functions defined using monotonic primitives. This later algorithm could be seen as an optimisation of the basic algorithm as it in general would require fewer iterations, but depending on the method used to represent pending environments the simpler algorithm may be better. The report failed to prove the basic algorithm correct for self-applicatory functions, even over the original binary domain; on the other hand it also failed to find any counter examples. Since then I have found a simple proof of the correctness of binary pending analysis, making use of the correctness of the "improved" algorithm. A similar proof works for all finite domains, but needs yet another incremental fixed point theorem! These proof are given below.

The proofs are split into two parts:

Optimality - the resulting function is no greater than the minimal fixed point

Safety - it is no lower

Safety is clearly most important for abstract interpretation, but is the more difficult. Optimality is proved in both the binary and finite domain case by direct comparison with the minimal fixed point. It is "obviously" true at one level, as the pending analysis makes values smaller, but it also makes the intermediate functionals non-monotonic whence nothing is obvious anymore.

Both optimality and safety proofs depend on the pseudo-monotonicity of the defining functional, that is, if $f \leq g$ and either $f$ or $g$ is monotonic then $G f \leq G g$. There are examples of monotonic functionals which are not pseudo-monotonic, for example:

$$G : (2 \rightarrow 2) \rightarrow (2 \rightarrow 2)$$

$$G f = \top \quad \text{if } f \text{ monotonic}$$
$$= \bot \quad \text{otherwise}$$

$G$ is a monotonic functional, with fixed point $f = \top$, unfortunately pending analysis yields the answer $f = \text{not}$, which is neither monotonic, fixed nor safe. However, any functional which is defined using terms and monotonic primatives is pseudo-monotonic. (That is all functions produced by programs.)

Because pseudo-monotonicity requires one of the functions to be monotonic, the proofs for pending analysis have to work by comparing its non-monotonic intermediate functions with other monotonic functions that have known properties. In the case of optimality this can be the fixed point itself, however more work is required for safety.

In addition to producing the new proofs for simple pending analysis, the fixed point properties of configurations and the non-deterministic algorithm are reconsidered. Corrections are given for slips in the proofs in YCS107 and an updated algorithm is given which reflects the techniques used in the proofs in this paper, relaxing monotonicity conditions.
Note: the user is assumed to have a copy of YCS107 as several definitions of operators etc. are not repeated here.

1. Optimality of binary and general pending analysis

Recall that standard binary pending analysis is the function \( X_{\{\}} \), where

\[
X_{\mathcal{A}}|_a = \begin{cases} \bot & a \in \mathcal{A} \\ (G X_{\mathcal{A} + \{a\}})|_a & \text{otherwise} \end{cases}
\]

Where \( G \) is the defining functional and \( \mathcal{A} \) is the set of application values that are "pending".

We want to prove that \( X_{\{\}} \leq \text{fix } G \), to do this we prove in general:

**Theorem** - optimality of binary pending analysis

\[
X_{\mathcal{A}} \leq \text{fix } G
\]

**Proof**

Induction on size \( \mathcal{A} \).

Base case \( \mathcal{A} \) complete

\[
X_{\mathcal{A}} = \bot \leq \text{fix } G
\]

\text{QED} base case

Inductive case.

Sub-cases

- case \( a \in \mathcal{A} \)
  \[
  X_{\mathcal{A}}|_a = \bot \leq \text{fix } G|_a
  \]
  \text{QED} \ a \in \mathcal{A}

- case \( a \notin \mathcal{A} \)
  \[
  X_{\mathcal{A}}|_a = G X_{\mathcal{A} + \{a\}}|_a
  \]
  but
  \[
  X_{\mathcal{A} + \{a\}} \leq \text{fix } G
  \]
  \Rightarrow \ G X_{\mathcal{A} + \{a\}} \leq G \ \text{fix } G = \ \text{fix } G
  \]
  \text{- induction}

\[
\text{QED} \ a \notin \mathcal{A}
\]

\text{QED} inductive cases

\text{QED} Theorem

Notice the pseudo-monotonicity step, that would not allow us for instance to prove \( X_{\mathcal{A} + \{a\}} \leq X_{\mathcal{A}} \).

The proof for general finite height domains is similar, except in that the definition involves a recursion:

\[
\begin{align*}
X_{\mathcal{C}}|_a &= \mathcal{C}(a) & a \in \text{dom } \mathcal{C} \\
X_{\mathcal{C}}|_a &= \text{fix } h & \text{otherwise}
\end{align*}
\]

where

\[
h(|d|) = (H X_{\mathcal{C} + \{a \rightarrow d\}})|_a
\]

Here \( H \) is used for the defining functional when the domain is non-binary. \( \mathcal{C} \) is a configuration, that is a partial function that represents the set of assumptions made about the target function during the course of the algorithm.

Note that \( h \) is in general not monotonic, so the standard meaning of fixed point does not apply. This point will be dealt with later, however in the mean time dangerous steps are marked !FIX!.

Again we want to prove that \( X_{\mathcal{C}} \leq \text{fix } H \). We need to include the obvious proviso that \( \mathcal{C} \ll \text{fix } H \).
Theorem - optimality of non-binary pending analysis

\[ C \ll \text{fix } H \Rightarrow X^C_L \leq \text{fix } H \]

Proof

Induction on size \( C \).

Base case \( C \) complete

\[ X^C_L = C \leq \text{fix } H \]

QED base case

Inductive case.

Sub-cases

\begin{itemize}
  \item \( a \in \text{dom } C \)
    \[ X^C_L |_a = C(a) \leq \text{fix } H \]
    QED \( a \in A \)
  \item \( a \notin \text{dom } C \)
    \[ X^C_L |_a = \text{fix } h \]
    \begin{align*}
      \text{where} & \quad h(d) = (H X^{C \uplus(a \rightarrow d)}_L |_a) \\
      \text{but} & \quad d \leq \text{fix } H \text{|}_a \\
      \Rightarrow & \quad C \uplus(a \rightarrow d) \ll \text{fix } H \\
      \Rightarrow & \quad X^{C \uplus(a \rightarrow d)}_L \leq \text{fix } H \quad \text{- induction} \\
      \Rightarrow & \quad H X^{C \uplus(a \rightarrow d)}_L \leq H \text{fix } H \quad \text{- pseudo-monotonicity} \\
      \Rightarrow & \quad h(d) \leq \text{fix } H \text{|}_a \\
      \therefore & \quad \text{fix } h \leq \text{fix } H \text{|}_a \quad \text{- !FIX!} \\
    \end{align*}
    QED \( a \notin A \)
\end{itemize}

QED inductive cases

QED

These two proofs are straight-forward and need none of the results from YCS107 except for the pseudo-monotonicity of functions defined by terms.

2. Safety of binary pending analysis

We want to prove that \( X_{\{\}} \geq \text{fix } G \), to do this we prove that in general:

\[ A \subseteq A \Rightarrow X_A \geq Z_A \]

\( Z_A \) is the improved algorithm defined by:

\[ Z_A |_a = \begin{cases} \\
\bot & a \in A \\
( G Z_{A \oplus \{a\}} ) |_a & \text{otherwise} \\
\end{cases} \]

where \( A \oplus \{a\} \) is \( A \) extended by \( a \) and then downwards closed. That is:

\[ A \oplus \{a\} = A \cup \{ a' | a' \leq a \} \]

In YCS107 we proved that \( Z_A = \text{fix } G_A \), where \( G_A \) is defined as follows:

\[ G_A |_a = \begin{cases} \\
\bot & a \in A \\
G |_a & \text{otherwise} \\
\end{cases} \]

So \( Z_{\{\}} = \text{fix } G_{\{\}} = \text{fix } G \). Thus the general result will be sufficient for safety of pending analysis.
Theorem - safety of binary pending analysis

\[ \mathcal{A} \subseteq \mathcal{A}' \implies X_\mathcal{A} \geq Z_{\mathcal{A}'} \]

Proof

Induction on size \( A \).

Base case \( A \) complete \( \implies A' \) complete

\[ X_\mathcal{A} = \perp = Z_{\mathcal{A}'} \]

QED base case

Inductive case.

Sub-cases

- case \( a \in \mathcal{A}' \)
  \[ X_{\mathcal{A}'} |_a \geq \perp = Z_{\mathcal{A}'} |_a \]
  QED \( a \in \mathcal{A} \)

- case \( a \notin \mathcal{A} \) \( \implies a \notin \mathcal{A} \)
  \[ X_{\mathcal{A}'} |_a = G X_{\mathcal{A}+\{a\}} |_a \]
  \[ Z_{\mathcal{A}'} |_a = G Z_{\mathcal{A} \oplus \{a\}} |_a \]
  but
  \[ \mathcal{A}+\{a\} \subseteq \mathcal{A}' \oplus \{a\} \]
  \[ \implies X_{\mathcal{A}+\{a\}} \geq Z_{\mathcal{A} \oplus \{a\}} \]
  \[ \implies G X_{\mathcal{A}+\{a\}} \geq G Z_{\mathcal{A} \oplus \{a\}} \]
  - induction
  - pseudo-monotonicity
  QED \( a \notin \mathcal{A} \)

QED inductive cases

QED

3. Safety of non-binary pending analysis

We might hope to prove a similar result for non-binary functions:

\[ \mathcal{C} \geq \mathcal{C}' \implies X^\mathcal{C} \geq X^\mathcal{C'} \]

Unfortunately this fails. If we follow the same proof procedure as for optimality we need to prove \( \text{fix } h \geq \text{fix } h' \) where:

\[ h( d ) = ( H X^\mathcal{C}_{\mathcal{A}+\{a\}} |_{a} ) |_{d} \]
\[ h'( d ) = ( H X^\mathcal{C'}_{\mathcal{A} \oplus \{a\}} |_{a} ) |_{d} \]

This is in general not true because \( H_\mathcal{C} \) can "hoist" up the value at \( a \) if \( \text{dom } \mathcal{C} \) contains elements comparable with \( a \).

We can proceed however by creating a new functional \( H^\mathcal{F}_\mathcal{C} \) which is monotonic and has an incremental fixed point theorem, but which doesn’t hoist up the target value. We will find we can prove that \( X^\mathcal{C} \geq \text{fix } H^\mathcal{F}_\mathcal{C} \)

The new functional is defined in a similar manner to \( H^\mathcal{C} \) and \( H_\mathcal{C} \) except with a different bounding functional \( \mathcal{F}_\mathcal{C} \):

\[ H^\mathcal{F}_\mathcal{C} = \mathcal{F}_\mathcal{C} H \]
\[ \mathcal{F}_\mathcal{C} I = \mathcal{C}^T \land I \]

That is \( \mathcal{F}_\mathcal{C} \) simply bounds above by \( \mathcal{C} \) whilst retaining monotonicity. Notice too that \( H^\mathcal{F}_{\{\} } = H_\mathcal{C} \), so that a safety comparison with \( H^\mathcal{F}_\mathcal{C} \) will prove safety of \( X^\{\} \). Note too that like \( H_\mathcal{C} \) (and unlike \( H^\mathcal{C} \)) \( H^\mathcal{F}_\mathcal{C} \) requires that the configuration \( \mathcal{C} \) be monotonic.

The incremental fixed point theorem for \( H^\mathcal{F}_\mathcal{C} \) merely added to the configuration \( (\mathcal{C} + (a \rightarrow d)) \), whereas \( H_\mathcal{C} \) needed a special augmentation operator \( (\mathcal{C} \oplus (a \rightarrow d)) \). The similar theorem for \( H^\mathcal{F}_\mathcal{C} \) needs an adding operator \( (\mathcal{C} \land (a \rightarrow d)) \) that bounds above but which preserves monotonicity:
\[ \text{dom} \left( \mathcal{C} \land (a \rightarrow d) \right) = \text{dom} \mathcal{C} + \{ a \} \]

\[ \mathcal{C} \land (a \rightarrow d) |_{a = a} = d \land \mathcal{C}'(a \rightarrow d) \]

\[ \mathcal{C} \land (a \rightarrow d) |_{a = a} = d \land C(\alpha') a' \leq a \]

Essentially \( \mathcal{C} \land (a \rightarrow d) \) is the biggest monotonic configuration less than both \( \mathcal{C} \) and \( \{ a \rightarrow d \} \).

The incremental fixed point theorem is familiar enough by now. Its proof is nearly identical in form to that for \( H_{\mathcal{C}} \) except that all the lemmas needed about \( B_{\mathcal{C}} \) are trivially true for \( F_{\mathcal{C}} \). We will label the steps as for the proof for \( H_{\mathcal{C}} \) to emphasise similarity.

**Theorem** - incremental fixed point theorem for \( H_{\mathcal{C}} \)

**Proof**

We now need to prove the opposite inequality:

(4) let \( I = \text{fix} H_{\mathcal{C}_{fix}} \)

prove that \( H_{\mathcal{C}} I = I \)

(4.1) \( H_{\mathcal{C}} I |_{a} = d_{\text{fix}} \)

immediate from def'n of \( g \) and \( g(d_{\text{fix}}) = d_{\text{fix}} \)

(4.2) \( \Rightarrow H_{\mathcal{C}_{fix}} I = H_{\mathcal{C}} I \)

as (4.1) \( \Rightarrow H_{\mathcal{C}} I \leq (\mathcal{C} \land (a \rightarrow d')) \top \)

(4.3) That is \( I = (H_{\mathcal{C}} I) \)

**QED** (4)

(5) thus \( \text{fix} H_{\mathcal{C}_{fix}} = I \geq \text{fix} H_{\mathcal{C}} \)

- minimality of \( \text{fix} H_{\mathcal{C}} \)

**QED** (ii)

(6) \( \text{fix} H_{\mathcal{C}} = H_{\mathcal{C}} (\text{fix} H_{\mathcal{C}_{fix}}) = H_{\mathcal{C}} (\text{fix} H_{\mathcal{C}_{fix}}) \)

\[ \Rightarrow \text{fix} H_{\mathcal{C}} |_{a} = g(d_{\text{fix}}) = d_{\text{fix}} \]

**QED** (i)

The final step is to prove the correctness of \( X_{\mathcal{C}} \) by comparison with \( \text{fix} H_{\mathcal{C}} \)

**Theorem** - correctness of non-binary pending analysis

\[ \mathcal{C} \geq \mathcal{C}' \Rightarrow X_{\mathcal{C}} \geq \text{fix} H_{\mathcal{C}} \]

**Proof**

Induction on size \( \mathcal{C} \).

Base case \( \mathcal{C} \) complete \( \Rightarrow \mathcal{C}' \) complete

\[ X_{\mathcal{C}} = \mathcal{C} \geq \mathcal{C}' \geq \text{fix} H_{\mathcal{C}} \]

**QED** base case
Inductive case.

Sub-cases

\[ X_{C}^{a} = C(a) \geq C'(a) \geq \text{fix } H_{C}' \]

QED \( a \in A \)

\[ X_{C}^{a} = \text{fix } h \]

where

\[ h(d) = (H X_{C'}^{a \rightarrow d})_{a} \]

but

\[ \forall d \quad X_{C'}^{a \rightarrow d} \geq \text{fix } H_{C}' \]

\[ \Rightarrow \quad H X_{C'}^{a \rightarrow d} \geq H \text{fix } H_{C}' \]

\[ \geq \quad H_{C}' \text{fix } H_{C}' \]

so

\[ h \geq g \quad \text{- where } g \text{ is the function in the fixed point theorem} \]

\[ \Rightarrow \quad \text{fix } h \geq \text{fix } g = (\text{fix } H_{C}')_{a} \quad \text{- !FIX!} \]

QED \( a \notin A \)

QED inductive cases

4. Errata - Fixed point properties of monotonic configurations

Note the proof given in YCS107 was intended for monotonic functionals and monotonic configurations. In fact as stated, it starts iterating at a non-monotonic configuration so works instead for fully-monotonic functionals over possibly non-monotonic configurations! The iteration in the proof should start at \( C \perp \), and is then correct.

5. Fixed point properties of non-monotonic configurations

We can produce a fixed point theorem for non-monotonic configurations, namely:

\[ C \Rightarrow H C \Rightarrow C \Rightarrow \text{fix } H \]

Proof
let \( f_0 = \bot \)
\[ f_{i+1} = H f_i \]
be the standard sequence to give \( \text{fix} \ H \).
We want to prove inductively that \( C \gg f_i \).

**base case**
\[ f_0 = \bot \text{ and } C \gg \bot \]
**QED**

**inductive case**
by inductive hypothesis \( C \gg f_i \)
\[ \Rightarrow H C \gg H f_i \] - pseudo-monotonicity
\[ \Rightarrow C \gg f_{i+1} \]
**QED**

So if we can construct (by some means!) a configuration with \( C \gg H C \) it is a safe (albeit non-monotonic) approximation to \( \text{fix} \ H \). If in addition one can show by some alternative argument that \( C \ll \text{fix} \ H \), then the configuration will be exactly \( \text{fix} \ H|_{\text{dom} \ C} \).

### 6. Errata - Non-deterministic algorithm using monotonic configurations

The definition of \( C \uparrow C' \) should read:

\[
\text{dom} (C \uparrow C') = \text{dom} C \cup \text{dom} C'
\]

\[
(C \uparrow C')_a = C^T l_a \lor C'^T l_a
\]

That is the least monotonic configuration greater than both \( C \) and \( C' \). Simple least upper bound is used below for the non-monotonic version.

### 7. Non-deterministic algorithm allowing non-monotonic configurations

With the above version of the fixed point theorem for non-monotonic configurations we can mirror the algorithm used in YCS107 to allow non-monotonic intermediate configurations. As with the case for using standard pending analysis as opposed to the version using monotonicity it depends on the usage and implemention which is better, but at least this allows the use of a simpler algorithm if desired.

The algorithm is nearly the same as the monotonic version, except that it uses simple configuration upper-bound \( (C \lor C') \) rather than the upward sum \( (C \uparrow C') \) defined above. \( C \lor C' \) is in fact the operator defined wrongly in YCS107 just to confuse things.

\[
\text{dom} (C \lor C') = \text{dom} C \cup \text{dom} C'
\]

\[
(C \lor C')_a = \bar{C}_a \lor \bar{C'}_a
\]

In addition this algorithm makes no monotonicity provision at step 3.

Define a sequence
let $C_i$ be a sequence of configurations defined thus:

$C_0$ any configuration such that $C_0 \ll \text{fix } H$

for each $i$ either

1. choose $a \in \text{dom } C, \ d \leq (\text{fix } H)_a$
   
   $C_{i+1} = C_i \lor (a \rightarrow d) \Rightarrow C_{i+1} \ll \text{fix } H$

2. choose $C \subseteq C_i$
   
   then choose $C' \subseteq H \ C$
   
   $C_{i+1} = C_i \lor C' \Rightarrow C_{i+1} \ll \text{fix } H$ - pseudo-monotonicity

3. choose any (possibly non-monotonic) $C_{i+1} \ll C_i$
   
   $C_{i+1} \ll \text{fix } H$

Obtain result

for any $i$ choose $C \subseteq C_i$ such that $C \gg H \ C$

$\Rightarrow C \subseteq \text{fix } H$

Again if step 3 is never used (and why should it be?) the algorithm is bound to terminate as the sequence $C_i$ is increasing. With suitable instantiation this furnishes an alternative proof of the various algorithms above. In addition it allows for optimised versions of the algorithm.

### 8. Meaning of fixed point for non-monotonic functions

For non-monotonic functions, we cannot use the standard fixed point algorithm without comment:

$$d_0 = \bot$$

$$d_{i+1} = f(d_i)$$

$$\text{fix } f = \lor d_i$$

We can construct the set $d_i$ as before, but it is not in general increasing, and the least upper bound need not be fixed. This is particularly nasty when viewed as an effective algorithm in a finite domain as the sequence $d_i$ need not stabilise, leading to possible non-termination. There are several definitions we can use to extend the operator fix to non-monotonic functions however:

(i) the definition used above

(ii) use $d_i$ as above but use $\lor \bigwedge_{n \in \text{dom }} d_i$

(iii) use $d_i$ as above but use $\bigwedge_{n \lor \bigvee d_i}$

(iv) define a new increasing sequence

$$d_0 = \bot$$

$$d_{i+1} = f(d_i) \lor d_i$$

$$\text{fix } f = \lor d_i$$

All these definitions are equivalent to standard fix on monotonic functions. (i), (ii) and (iii) all share the termination problem, so an alternative proof of termination is necessary. (iv) is an increasing sequence, so is bound to terminate. Note that using (iv) we have $f(\text{fix } f) \leq \text{fix } f$. The other definitions have no such nice properties. It is not necessarily true that (iv) ascends faster than the rest, although it looks as though it should. However in non-perverse examples it probably will "on average"!

Two properties are used of fix in the above theorems, both are true for all definitions:

1. $(d \leq d' \Rightarrow f(d) \leq d') \Rightarrow \text{fix } f \leq d'$

2. fix is pseudo monotonic, that is:
\[ f \leq g \text{ and one of } f \text{ or } g \text{ is monotonic} \]
\[ \Rightarrow \text{fix } f \leq \text{fix } g \]

The former is a simple induction on \(d_i\). The pseudo-monotonicity is only slightly more subtle,

**Proof** - pseudo-monotonicity of fix (all defn's)

- let \(d_i\) be the sequence for \(f\) and \(c_i\) be the sequence for \(g\)
  (using whichever definition we are interested in)
- we want to show by induction that \(d_i \leq c_i\)

  **base case**
  \[d_0 = \bot = c_0\]

  **QED** base case

  **inductive case**
  by inductive hypothesis \(d_i \leq c_i\)

  if \(f\) is monotonic then
  \[f(d_i) \leq f(c_i) \leq g(c_i)\]

  otherwise \(g\) is monotonic and
  \[f(d_i) \leq g(d_i) \leq g(c_i)\]

  either way \(f(d_i) \leq g(c_i)\) whence by whichever definition we use we get \(d_{i+1} \leq c_{i+1}\)

  **QED** inductive case

  in each case the construction of fix \(f\) from \(d_i\) uses standard operations, so fix \(f \leq \text{fix } g\)

  **QED** theorem

9. Conclusions

If I had this proof before writing YCS107, it would have been shorter and simpler. However, it would have been a pity not to have the incremental fixed point theorem for \(\mathbb{R}_e\), and in general to have missed the breadth and depth of analysis. Also I would have been unlikely to have produced the non-deterministic algorithm which promises far more power and flexibility than the other algorithms considered.

It is very nice to finally have no loose ends dangling.
## Appendix - Notation index

The following is an index to most of the notation used in this report and YCS107.

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</tr>
<tr>
<td>( \mathcal{C} \lor \mathcal{C} )</td>
<td>here</td>
<td>§8</td>
<td>simple symmetric, not necessarily monotonic least upper bound operator for configurations</td>
</tr>
<tr>
<td>( \text{fix} )</td>
<td>here</td>
<td>§9</td>
<td>various generalisations of the fixed point operator for use with non-monotonic functions</td>
</tr>
</tbody>
</table>